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Diffusion Dynamics of Classical Systems Driven by an Oscillatory Force

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We investigate the asymptotic behavior of solutions to a kinetic equation describing the evolution of particles subject to the sum of a fixed, confining, Hamiltonian, and a small time-oscillating perturbation. Additionally, the equation involves an interaction operator which projects the distribution function onto functions of the fixed Hamiltonian. The paper aims at providing a classical counterpart to the derivation of rate equations from the atomic Bloch equations. Here, the homogenization procedure leads to a diffusion equation in the energy variable. The presence of the interaction operator regularizes the limit process and leads to finite diffusion coefficients.

KEY WORDS: Kinetic equation, homogenization, diffusion limit.
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1. SETTING OF THE PROBLEM

We consider a particle system described by its phase-space density, or distribution function, $f(t, x, p)$: $x \in \mathbb{R}^d$ is the position variable, $p \in \mathbb{R}^d$ is the momentum, and t is the time. In practice, $d = 1, 2$ or 3. It is convenient to also introduce the phase space variable $X = (x, p) \in \mathbb{R}^{2d}$. The evolution of the density f is governed by a kinetic equation of the form

$$\partial_t f + \{H, f\} = \frac{1}{\tau} Q(f). \tag{1.1}$$

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Given the Hamiltonian of the system $H = H(t, X) = H(t, x, p)$, the Poisson bracket $\{H, f\}$ denotes the operator

$$\{H, f\} = \nabla_p H \cdot \nabla_x f - \nabla_x H \cdot \nabla_p f.$$

The left-hand side of (1.1) describes the total time derivative of f along the trajectories of the particles, i.e.

$$\frac{d}{dt} f(t, \bar{x}(t), \bar{p}(t)) = (\partial_t f + \{H, f\})(t, \bar{x}(t), \bar{p}(t)),$$

where $\overline{X}(t) = (\bar{x}(t), \bar{p}(t))$ is any solution of the characteristic system

$$\frac{d}{dt}\bar{x}(t) = \nabla_p H(t, \bar{x}(t), \bar{p}(t)), \quad \frac{d}{dt}\bar{p}(t) = -\nabla_x H(t, \bar{x}(t), \bar{p}(t)).$$

Then, (1.1) translates the fact that the time variations of f produced by transport along the Hamiltonian flow of H balances the rate of change of f. The latter is due to complex interaction phenomena, the description of which is embodied into the operator Q (see below). The parameter $\tau > 0$ in (1.1) then appears as a relaxation time.

We are interested in a situation in which the Hamiltonian H splits into an unperturbed time-independent Hamiltonian $H_0(x, p)$, and a time dependent potential perturbation $\mathcal{V}(t, x)$, i.e.

$$H(t, x, p) = H_0(x, p) + \mathcal{V}(t, x).$$

The technical requirements on H_0 and $\mathcal{V}$ will be specified later on. A typical example is that of a classical particle in an unperturbed potential $V_0(x)$ which leads to

$$H_0(X) = \frac{p^2}{2} + V_0(x).$$

The prototype situation is the case where H_0 is the harmonic oscillator

$$H_0(X) = \frac{p^2 + x^2}{2} = H_{\text{harm}}(X).$$

This situation is presented in detail in Appendix E.1.

Besides, we assume that the potential $\mathcal{V}$ is small but has very fast time variations. Precisely, let us denote by ε the ratio between the order of magnitude of the perturbation and that of the free Hamiltonian. We also have to define the observation time scale T, in comparison to both the typical time scale of the perturbation θ and the relaxation time τ. It turns out that the perturbation is still negligible when looking at too short time scales (say of order $\mathcal{O}(1/\varepsilon)$). This is reminiscent of the well established fact that perturbations of size ε in an integrable Hamiltonian dynamics enter at second order only: they induce an effect of typical size $O(\varepsilon^2)$. In this paper, the "integrability" assumption is played by Hypothesis 1.2

below. For that reason, we define the time scale so that $\mathrm{T}/\theta = 1/\varepsilon^2$, $\mathrm{T}/\tau = \gamma/\varepsilon^2$, with $\gamma > 0$ a fixed dimensionless parameter. Accordingly, the Hamiltonian can be recast in dimensionless form as

$$H(t, x, p) = H_0(x, p) + \varepsilon V(t/\varepsilon^2, x)$$

and we wish to perform the asymptotic analysis $\varepsilon \to 0$ in the following scaled version of (1.1)

$$\varepsilon^2\, \partial_t f^\varepsilon + \left\{H_0, f^\varepsilon\right\} + \varepsilon \left\{V(t/\varepsilon^2, x), f^\varepsilon\right\} = \gamma\ Q(f^\varepsilon). \tag{1.2}$$

The derivation of (1.2) from (1.1) is detailed in Appendix B.1. Such a scaling is known under the name of weak-coupling regime, and is a well-identified regime both in quantum mechanics and in classical Hamiltonian systems.[36]

The present situation is the standard setting for the description of an atom which interacts with a light field. In that case, the unperturbed Hamiltonian H_0 is the atomic Hamiltonian, and the perturbation $\mathcal{V}(t, x) = \varepsilon V(t/\varepsilon^2, x)$ is the potential energy induced by the light wave in the vicinity of the atom. If a quantum mechanical setting is retained instead of a classical one, the kinetic equation (1.1) must be replaced by the quantum Liouville equation, which, for atoms, is often referred to as the atomic Bloch equation. It reads

$$i\varepsilon^2 \partial_t \rho^\varepsilon(t) = \left[H_0, \rho^\varepsilon(t)\right] + \varepsilon \left[V(t/\varepsilon^2), \rho^\varepsilon(t)\right] + \gamma\ Q(\rho^\varepsilon(t)), \tag{1.3}$$

where the unknown now is a time dependent trace class operator $\rho^\varepsilon(t)$, the so-called density matrix of the quantum mechanical system, and all Poisson brackets $\{\cdot, \cdot\}$ are formally replaced by commutators $[\cdot, \cdot]$ between operators, in the passage from the kinetic equation (1.2) to the quantum equation (1.3). Also, in (1.3), $Q(\rho^\varepsilon)$ is a relaxation operator that describes, at a heuristic level, the observed trend of various atomic systems to relax towards equilibrium states of the unperturbed Hamiltonian H_0. We do not give the precise expression of $Q(\rho^\varepsilon)$ here, and refer e.g. to Ref. (27) for a physical discussion.

Let us now turn to the definition of the operator Q that is relevant in our context. Our basic approach follows the analogy between the quantum mechanical situation (1.3) and the associated classical setting (1.2). For quantum mechanical systems, the large time behavior of the system can be described by a time-differential system of rate equations, which describes the evolution of the populations of the atomic energy levels (see e.g. Ref. (27) and references therein). The rate constants depend on the frequency of the light field and the differences between the atomic energy levels (transition energies). They are large when a resonance occurs i.e. when the frequency of the light field matches one (or more) of the transition energies. These facts have been recently proved on a rigorous basis in Refs. (8, 9), starting from equation (1.3) and performing both a density matrix analysis in the spirit of Refs. (14, 12, 13), and an averaging procedure for Ordinary Differential Equations in the spirit of Ref. (35). In the present work, we would like to explore a similar situation with a classical system. The classical counterpart of the level population is the number of particles on a given energy surface. Hence, we shall assume that this number is well defined and finite for almost all energies. For that purpose, let us introduce the following requirements on the free Hamiltonian H_0.

Hypothesis 1.1. *We assume that*

$$H_0(X) \in C^\infty(\mathbb{R}^{2d}), \quad H_0(X) \geq -C_0 \quad \text{for some } C_0 \geq 0, \quad \lim_{|X|\to\infty} H_0(X) = +\infty.$$

Hypothesis 1.2 (Well defined energy levels, having finite measure). *We assume that*

(i) For almost all $E \in \mathbb{R}$, the set[4]

$$S_E = \{X = (x, p) \in \mathbb{R}^{2d} \mid H_0(X) = E\},$$

is a smooth orientable $2d - 1$ submanifold of $\mathbb{R}^{2d}$. For any such E, we let $d\sigma_E(X)$ denote the induced euclidean surface measure, and we define the measure $\delta(H_0(X) - E)$ as

$$\delta(H_0(X) - E) := \frac{d\sigma_E(X)}{\left|\nabla_X H_0(X)\right|}.$$

(ii) For any E as in (i), S_E also has finite measure with respect to $\delta(H_0(X) - E)$. In other words

$$h_0(E) := \int_{S_E} \delta(H_0(X) - E) < +\infty, \quad a.e.\ E \in \mathbb{R}.$$

This serves as a definition for $h_0(E)$.

Hypothesis 1.3. *Let $\overline{X} : s \in \mathbb{R} \longmapsto \overline{X}(s) \in \mathbb{R}^{2d}$ stand for the solution of the ODE system*

$$\frac{d}{ds}\overline{X}(s) = (\nabla_p H_0, -\nabla_x H_0)(\overline{X}(s)), \quad \overline{X}(0) = (x, p).$$

Then we assume that the matrix of the derivatives with respect to the initial data is such that for any $0 < R < \infty$, there exist $C_R, q_R \geq 0$ verifying

$$\sup_{|(x.p)| \leq R} |\nabla_{x,p}\overline{X}(s)| \leq C_R\,(1 + |s|)^{q_R}$$

for any $s \in \mathbb{R}$.

[4] We should write here $E \in H_0(\mathbb{R}^{2d})$ instead of $E \in \mathbb{R}$ to be rigorous. Since the distinction between $H_0(\mathbb{R}^{2d})$ and $\mathbb{R}$ is anyhow obvious – there is nothing to assume for energies $E \notin H_0(\mathbb{R}^{2d})$ – we shall systematically consider energies $E \in \mathbb{R}$ in this article, meaning implicitely that energies should actually satisfy the rigorous condition $E \in H_0(\mathbb{R}^{2d})$.

Remark 1.1. *Of course, these assumptions are fulfilled by the harmonic potential* H_{harm}*. Then, the energy shells reduce to spheres* $\{X \in \mathbb{R}^{2d},\ X^2 = 2E\}$ *and Hypothesis (1.3) simply holds with* $C_R = 1$*,* $q_R = 0$*. Moreover, one may take any smooth diffeomorphism of phase-space* $\Phi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$*. Clearly, the new Hamiltonian* $H_0(X) = H_{\text{harm}}(\Phi(X))$ *also satisfies these Hypotheses. Then, energy shells are deformed spheres.*

Remark 1.2. *Hypothesis (1.1) is essentially a confining condition. As discussed in Appendix A.1, once* H_0 *is assumed* C^∞*, Sard's Theorem together with the coarea formula imply that* S_E *is indeed a smooth codimension one submanifold, for almost every* $E \in \mathbb{R}$*. Hence part (i) of Hypothesis (1.2) is indeed a consequence of Hypothesis (1.1) The important point in Hypothesis (1.2) is part (ii). It can be seen as an additional growth condition on* H_0 *with respect to the space variable. It allows us to normalize the measure* $\delta(H_0(x) - E)$*. This is a key assumption in the present paper, both from the point of view of the model (it allows us to define the operator Q), and of the techniques: through Jensen's inequality, it gives us the desired "entropy estimates" suited for our asymptotic analysis. Note that the measure* $\delta(H_0(x) - E)$ *is a standard object in statistical physics: it is known as the microcanonical measure on the energy shell* $S_E = \{H_0(X) = E\}$*. It is also refered to as the Liouville measure, which is the unique invariant measure under the Hamiltonian flow generated by* H_0*.*

Also, Hypothesis 1.3 is a strong stability assumption on the unperturbed potential V_0*. Its role will appear clear in Section 4.2, and is related to the regularity of the solutions of certain profile equations. Note that this Hypothesis can be relaxed, but at the price of restricting the relaxation parameter* γ *to large enough values.*

Associated with $\delta(H_0(X) - E)$, the following mean-value operator is defined:

$$\Pi f(t, E) := \frac{1}{h_0(E)} \int_{S_E} f(t, X)\, \delta(H_0(X) - E) = \frac{\displaystyle\int_{S_E} f(t, X)\, \delta(H_0(X) - E)}{\displaystyle\int_{S_E} \delta(H_0(X) - E)}. \tag{1.4}$$

For each energy level E, Πf defines the average of f over the energy shell $\{X \mid H_0(X) = E\}$. In Appendix A.1, we check that $\Pi f(t, E)$ is well-defined for functions f belonging to the spaces $L^p(\mathbb{R}^{2d})$. Physically, $\Pi f(t, E)$ denotes the mean number of particles which belong to the energy shell S_E at time t. Now, the classical counterpart of the level populations being the number of particles on a given energy surface, it is natural to define the following operator

$$P : f \longmapsto Pf(t, X) := \Pi f(t, H_0(X)). \tag{1.5}$$

We shall see that P enjoys the natural self-adjointness and contraction properties of a projection: it is the projection onto functions depending only on the energy.

Going on with the analogy between classical and quantum mechanics, we also observe that the classical counterpart of the density-matrix correlations is the projection of the distribution function onto the space orthogonal to functions of the energy only. This leads us to the following definition of the relaxation operator to be used in (1.2):

$$Q(f) := Pf - f. \tag{1.6}$$

This operator models the relaxation of the distribution function towards a function of the total energy of the system only. Physically, it describes a redistribution of the particles which makes the distribution uniform on any energy shell. To motivate this interaction, we can think of some resonant interaction process: two particles with different energies do not spend enough time in a coherent motion one with respect to each other to interact significantly. Only particles which have the same energy do interact, and if this interaction is repulsive, it eventually produces a uniform distribution on the energy shell. Further considerations on how such a relaxation operator can be derived are beyond the scope of this work.

Let us give some intuition of the phenomena involved in (1.2), endowed with the operator (1.6). First, as $\varepsilon \to 0$, we can expect that f^ε relaxes towards an equidistributed repartition i.e. towards a solution to $Pf = f$. However, the fluctuations $f^\varepsilon - Pf^\varepsilon$, which are small but definitely non zero, are transported by the Hamiltonian flow. Then, resonant interactions are possible with the motion induced by the perturbation εV which oscillates with frequency $1/\varepsilon^2$. These intricate interactions will eventually give rise to diffusion in the energy variable. Of course, the asymptotics is highly governed by the precise time dependence of V. It turns out that the relaxation operator Q somewhat regularizes the situation in this respect, in that it prevents the possibility of too strong resonances (small denominators), through the introduction of some damping in the model. Let us comment further the introduction of this operator:

- On the one hand, as explained above, the situation has to be compared with the quantum Bloch equation (1.3), which has been analyzed in Ref. (8) and further in Ref. (9). There, the term $Q(\rho^\varepsilon)$ gives damping terms for the off-diagonal elements of the density matrix (the correlations, analogous to $f^\varepsilon - Pf^\varepsilon$ here). These damping terms make the large-time dynamics dominated by the diagonal elements (the populations, analogous to Pf^ε here). They also contribute to making the rate constants finite even at resonances (the "width" of the resonance being related to the damping rates). These damping terms can be physically motivated in a number of ways (for instance they can model the decoherence effects of atomic collisions in a gaseous medium, see the discussion in Ref. (27)). Under more restrictive assumptions on the data, smaller damping rates of order $\mathcal{O}(\varepsilon^\mu)$ with $\mu < 1/2$ could be considered and the usual (undamped) formulae for the Einstein rate equations[27] could be recovered.[8,9]

- On the other hand, the operator Q introduces non reversibility in the system through dissipation mechanisms. Without damping rates, the Bloch equation is time-reversible while the rate equations are time-irreversible. The damping terms in the quantum Liouville equation make it an irreversible equation from the beginning and simplifies the mathematical theory. A similar idea was used in Refs. (12, 13, 14) for the derivation of the Pauli master equation from the quantum Liouville equation in a deterministic framework. Indeed, it is a well-known fact, since the work of Lanford[(25)] about the derivation of the Boltzmann equation, that rigorously passing from a reversible to an irreversible dynamics is extremely difficult. A second, probably more standard, approach to overcome this problem is the introduction of stochastic averaging in the model, as in Ref. (16, 17, 26, 33) (see also Ref. (24) in a different context). There are several other examples of such an alternative: homogenization of convection(-diffusion) equations (see Refs. (22, 23) and references therein), Lorentz gas evolving in a billiard (see Refs. (7, 10)), quantum scattering limit of the Schrödinger equation.[(5,17,31,33)] For the (space-)homogenization of the kinetic equation without dissipative term, we refer e.g. to Refs. (2, 20). Here, as well as in Refs. (8, 9), we wish to treat the problem in a fully deterministic framework. To some extent, in this framework, the damping term plays the same role as the stochastic averaging process (see Remark 3.2 below).

We wish to add a last comment. In the quantum context, it has been proved (see Refs. (8, 9) for extensions) that the asymptotic behavior of the Bloch equations (1.3) leads to an Ordinary Differential System (the system of rate equations) describing the occupation numbers of the various energy levels. This system describes the jump process of the electrons between the energy levels. However, in contrast with the quantum case where the energy levels are naturally discrete (like the lowest energy levels of an atom), a classical system possesses a continuum of allowed energies and the corresponding transition energies are infinitesimaly small. Therefore, the large time evolution of a classical system (or equivalently, in our framework, the $\varepsilon \to 0$ limit of Eqs. (1.2), (1.6)) is expected to take place through infinitesimal energy jumps, i.e. through a diffusion process in energy, rather than through a finite jump process. For this reason, the limit model will be in the form of a diffusion equation in the energy variable, or in other words, of a Fokker-Planck type equation. The goal of the paper is to rigorously show this fact and to obtain the classical mechanics counterparts of the results proved in Refs. (8). The main result of this work can be summarized as follows.

Formal statement. *We suppose that V oscillates quasi-periodically: $V(\tau, x) = V_{\mathrm{q}}(\omega\tau, x)$, where $\omega \in \mathbb{R}^r$ has rationaly independent components and $\theta \mapsto V_{\mathrm{q}}(\theta, x)$ is $(0,1)^r$-periodic. Then, up to some "reasonable" assumptions on V_{q}, $f^{\varepsilon}(t, X)$ converges to some $F(t, H_0(X))$, where $F(t, E)$ satisfies a diffusion equation, which* *can be written in the following conservative form*

$$\partial_t(h_0 F) - \partial_E(h_0 b\, \partial_E F) = 0, \tag{1.7}$$

with h_0 defined in Hypothesis (1.2). The coefficient $b(E) \geq 0$ is defined by an expression involving some average of V_{q}.

The expression of the effective coefficient b, as well as the precise notion of convergence will be stated later on (see Section 3). In (1.7), $h_0 F(E)\, dE$ can be interpreted as the number of particles having their energies in the interval $(E, E + dE)$ while $h_0 b\, \partial_E F(E)$ gives the particle flux through the energy surface S_E.

The remainder of this paper is organized as follows. Section 2 is devoted to the basic properties of both the relaxation and transport operators, which will be crucial for our analysis. In Section 3, we provide a formal derivation of the asymptotic model. To this aim, we restrict ourselves to the framework of quasi-periodic perturbation potentials V. In this framework, we are able to give the precise and complete statement of our convergence result. This discussion allows us to point out the mathematical difficulties related to the resolution of adequate profile equations. These difficulties are analyzed in Section 4. Next, details of the convergence proof are presented in Section 5. We postpone the proofs of several technical facts – which could be interesting in themselves – to the Appendix.

2. PRELIMINARY CONSIDERATIONS: PROPERTIES OF THE RELAXATION OPERATOR

Since equations (1.2), (1.6) describe a relaxation phenomenon, we are naturally led to investigate the dissipation properties of the operator Q. This will give a particular form of the "entropy dissipation estimates" that are suited to our problem. Also, the commutator between both operators $f \longmapsto Pf$ and $f \longmapsto \{H_0, f\}$ is an important object in the asymptotic analysis of (1.2). Hence, the following statement will be useful.

Lemma 2.1. *The operator P satisfies the following properties:*

(i) P is a continuous projection operator on L^p spaces:

$$P(Pf) = Pf, \quad \|Pf\|_{L^p(\mathbb{R}^{2d})} \leq \|f\|_{L^p(\mathbb{R}^{2d})} \quad 1 \leq p \leq \infty.$$

(ii) P is conservative in the sense that for any integrable function, we get

$$\int_{\mathbb{R}^{2d}} Pf\, dX = \int_{\mathbb{R}^{2d}} f\, dX.$$

(iii) P is self-adjoint with respect to the inner product in $L^2(\mathbb{R}^{2d})$ (denoted by $\langle\cdot,\cdot\rangle$ throughout the paper). Consequently, the following orthogonality property holds: for any function $f \in L^2(\mathbb{R}^{2d})$ and $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$X \longmapsto \varphi(H_0(X))$ lies in $L^2(\mathbb{R}^{2d})$, we have

$$\langle \varphi(H_0(X)), (\mathrm{Id} - P)f \rangle = 0.$$

(iv) P is a non negative operator: if $f \geq 0$ almost everywhere (a.e.), then $Pf \geq 0$ a.e. as well. Moreover, the stronger property holds:

$$\text{If } f \geq 0 \text{ a.e., and } Pf = 0 \text{ a.e., then } f = 0 \text{ a.e.}$$

(v) The operators $f \longmapsto Pf$ and $f \longmapsto \{H_0, f\}$ are orthogonal, in the sense that

$$P\{H_0, f\} = 0,$$

holds for any $f \in L^2(\mathbb{R}^{2d})$ such that $\{H_0, f\} \in L^2(\mathbb{R}^{2d})$. Consequently, for any $f, g \in L^2(\mathbb{R}^{2d})$ such that $\{H_0, f\}$ and $\{H_0, g\}$ in $L^2(\mathbb{R}^{2d})$, we have

$$P(\{H_0, f\}g) = -P(f\{H_0, g\}).$$

Property (iii) implies that

$$\int_{\mathbb{R}^{2d}} (Pf - f)Pf \, dX = 0.$$

Therefore, we deduce the following key property of the relaxation operator.

Corollary 2.2. *The operator Q is a bounded operator on $L^2(\mathbb{R}^{2d})$ and the relation*

$$-\int_{\mathbb{R}^{2d}} Q(f)f \, dX = \int_{\mathbb{R}^{2d}} |Pf - f|^2 \, dX \geq 0$$

holds for any $f \in L^2(\mathbb{R}^{2d})$.

Proof of Lemma 2.1. We split the proof as follows.

Proof of (i)–(ii)–(iii)

The continuity of P on L^p spaces is an immediate consequence of the coarea formula recalled in Appendix A.1, together with the assumption that S_E has finite measure for $E \in \mathbb{R}$ a.e. Indeed,

$$\begin{aligned}
\|Pf\|^p_{L^p(dX)} &= \int_{\mathbb{R}^{2d}} \left|\Pi f(H_0(X))\right|^p dX \\
&= \int_{\mathbb{R}} \left|\Pi f(E)\right|^p h_0(E)\, dE \\
&\leq \int_{\mathbb{R}} \left(\int_{S_E} |f(X)|^p \, \frac{\delta(H_0(X) - E)}{h_0(E)} \right) h_0(E)\, dE \\
&\leq \int_{\mathbb{R}^{2d}} |f(X)|^p \, dX
\end{aligned}$$

where the coarea formula (A.5) is used for the second equality, Jensen's inequality for the first inequality and the coarea formula again for the second inequality. Note that equality holds for $p = 1$. The relation $P(Pf) = Pf$ is obvious since P leaves any function depending only on $H_0(X)$ invariant. Finally, the self-adjointness of P simply comes from the identity $P = \Pi^*\Pi$, where Π^* is the adjoint of Π (with the notations of the Appendix – see Lemma A.1.1).

Proof of (iv)
It is obvious that P preserves non negativeness. Let $f \geq 0$ such that $Pf = 0$ a.e.. Since $\int_{\mathbb{R}^{2d}} f\, dX = \int_{\mathbb{R}^{2d}} Pf\, dX = 0$, then, f is a nonnegative function with vanishing integral, which implies that $f(X) = 0$ for $X \in \mathbb{R}^{2d}$ a.e.

Proof of (v)
We deduce that $P\{H_0, f\} = 0$ from $\Pi\{H_0, f\} = 0$. To prove the latter, we take any test function $\psi(E) \in L^2(\mathbb{R}, h_0(E)\, dE)$. We write

$$\begin{aligned}
\left\langle \Pi\{H_0, f\}, \psi \right\rangle_{L^2(\mathbb{R}; h_0(E)dE)} &= \int_{\mathbb{R}} \Pi\{H_0, f\}(E)\, \psi(E)\, h_0(E)\, dE \\
&= \left\langle \left\{H_0, f\right\}, \Pi^* \psi \right\rangle_{L^2(\mathbb{R}^{2d})} \\
&= \left\langle \left\{H_0, f\right\}, \psi(H_0(X)) \right\rangle_{L^2(\mathbb{R}^{2d})} \\
&= -\left\langle f, \left\{H_0, \psi(H_0(X))\right\} \right\rangle_{L^2(\mathbb{R}^{2d})} \\
&= 0.
\end{aligned}$$

where the definition of Π^* can be found in Lemma A.1.1 of the Appendix and where we have used an integration by parts to obtain the fourth equality. Then, combining this property together with the Leibniz rule $\{H_0, f\, g\} = \{H_0, f\}\, g + f\, \{H_0, g\}$ allows to conclude the proof. ∎

3. FORMAL DERIVATION; QUASI-PERIODICITY

We consider a perturbation V which oscillates in a quasi-periodic way. To be more precise, let $\mathbb{Y}$ be the unit cube in $\mathbb{R}^r$, for some integer $r \geq 1$. We assume the following

Quasi-periodicity Hypothesis: *There exists a vector* $\omega \in \mathbb{R}^r \setminus \{0\}$ *and a smooth and bounded function* $V_{\mathrm{q}} : \mathbb{R}^r \times \mathbb{R}^d \to \mathbb{R}$, *which is* $\mathbb{Y}$*-periodic with respect to its first variable, such that*

$$V(\tau, x) = V_{\mathrm{q}}(\omega\tau, x), \quad \textit{for any } \tau \in \mathbb{R},\ x \in \mathbb{R}^d .$$

The periodicity condition means that $V_{\mathrm{q}}(\theta + j, x) = V_{\mathrm{q}}(\theta, x)$ holds for any $\theta \in \mathbb{Y}$, $x \in \mathbb{R}^d$, $j \in \mathbb{N}^r$. The vector ω is called the frequency vector. It collects the r

frequencies of V. We assume that the r components of ω are rationally independent, which means that $k \cdot \omega = 0$, for $k \in \mathbb{Q}^r$ iff $k = 0$. When $r = 1$, V is simply said to be periodic, and one can take $\omega = 1$ without loss of generality. It will be convenient later to make use of the Fourier series associated to V_{q}

$$V_{\mathrm{q}}(\theta, x) = \sum_{k \in \mathbb{Z}^r} \widehat{V}_{\mathrm{q}}(k, x)\ \exp(2i\pi k \cdot \theta),$$

$$\widehat{V}_{\mathrm{q}}(k, x) = \int_{\mathbb{Y}} V_{\mathrm{q}}(\theta, x)\ \exp(-2i\pi k \cdot \theta)\, d\theta.$$

Provided V_{q} has the smoothness $V_{\mathrm{q}}(\theta, x) \in L^2(\mathbb{Y} \times \mathbb{R}^d)$, the above Fourier series is convergent in the topology $\ell^2(\mathbb{Z}^r; L^2(\mathbb{R}^d))$ (note that we shall need the stronger regularity $V_{\mathrm{q}} \in C_b^2$, see Assumption 3.1 below).

With the help of this assumption, we can now guess the behavior of f^ε by inserting into Eq. (1.2) a double scale ansatz in the spirit of Ref. (6, 34):

$$f^\varepsilon(t, X) = f_{\mathrm{q}}^{(0)}(t, \omega t/\varepsilon^2, X) + \varepsilon f_{\mathrm{q}}^{(1)}(t, \omega t/\varepsilon^2, X) + \varepsilon^2 f_{\mathrm{q}}^{(2)}(t, \omega t/\varepsilon^2, X) + \cdots$$

where all functions $f_{\mathrm{q}}^{(i)}$ are supposed $\mathbb{Y}$-periodic with respect to the second variable. Then, we formally identify all terms which appear with the same power of ε. Remarking that

$$\partial_t \left(f_{\mathrm{q}}^{(i)}(t, \omega t/\varepsilon^2, X) \right) = \left(\partial_t f_{\mathrm{q}}^{(i)} + \frac{1}{\varepsilon^2}\, \omega \cdot \nabla_\theta f_{\mathrm{q}}^{(i)} \right) (t, \omega t/\varepsilon^2, X),$$

it becomes convenient to introduce the operator

$$\mathcal{T} f_{\mathrm{q}} = \omega \cdot \nabla_\theta f_{\mathrm{q}} + \{H_0, f_{\mathrm{q}}\} - \gamma Q(f_{\mathrm{q}}),$$

and its formal adjoint $\mathcal{T}^* \varphi = -\omega \cdot \nabla_\theta \varphi - \{H_0, \varphi\} - \gamma Q(\varphi)$. We obtain the following profile equations

$$\varepsilon^0 \text{ term:} \quad \mathcal{T} f_{\mathrm{q}}^{(0)} = 0, \tag{3.1}$$

$$\varepsilon^1 \text{ term:} \quad \mathcal{T} f_{\mathrm{q}}^{(1)} = \nabla_x V_{\mathrm{q}}(\theta, x) \cdot \nabla_p f_{\mathrm{q}}^{(0)}, \tag{3.2}$$

$$\varepsilon^2 \text{ term:} \quad \mathcal{T} f_{\mathrm{q}}^{(2)} = -\partial_t f_{\mathrm{q}}^{(0)} + \nabla_x V_{\mathrm{q}}(\theta, x) \cdot \nabla_p f_{\mathrm{q}}^{(1)} \tag{3.3}$$

and so on. The general form of these equation reads $\mathcal{T} f_{\mathrm{q}} = h_{\mathrm{q}}$, and the time variable t appears only as a parameter. As a matter of fact, we readily check that any function depending only on the energy variable, but not on θ, belongs to the kernel of $\mathcal{T}$. Therefore, it is tempting to infer from (3.1) that

$$f_{\mathrm{q}}^{(0)}(t, \theta, X) = F(t, H_0(X)).$$

Since such a function also lies in the kernel of the adjoint operator $\mathcal{T}^*$, we might imagine that the orthogonality relation

$$\int_{\mathbb{Y}} Ph_{\mathrm{q}}\, d\theta = 0$$

can serve as a compatibility condition. Assuming that these considerations hold true, and forgetting for the time being any functional difficulties, we rewrite (3.2) as

$$\mathcal{T} f_{\mathrm{q}}^{(1)} = \nabla_x V_{\mathrm{q}}(\theta, x) \cdot \nabla_p H_0(X)\, \partial_E F(t, H_0(X)).$$

Note that $\nabla_x V_{\mathrm{q}}(\theta, x) \cdot \nabla_p H_0(X) = -\{V_{\mathrm{q}}, H_0\}$ fulfils the compatibility condition, thanks to Lemma 2.1-(v). Thus, we can define $\chi_{\mathrm{q}}(\theta, X)$, a solution of the auxiliary equation

$$\mathcal{T} \chi_{\mathrm{q}} = \nabla_x V_{\mathrm{q}}(\theta, x) \cdot \nabla_p H_0(X),$$

and we set $f_{\mathrm{q}}^{(1)}(t, \theta, X) = \chi_{\mathrm{q}}(\theta, X)\partial_E F(t, H_0(X))$. Inserting this expression into the ε^2 order equation (3.3), and using the compatibility condition, we are led to

$$\begin{aligned}
0 &= \partial_t P(F(t, H_0(X))) - \int_{\mathbb{Y}} P\big(\nabla_x V_{\mathrm{q}}(\theta, x) \cdot \nabla_p f_{\mathrm{q}}^{(1)}(t, \theta, X)\big)\, d\theta \\
&= \partial_t F(t, H_0(X)) - \left(\int_{\mathbb{Y}} P\big(\nabla_x V_{\mathrm{q}}(\theta, x) \cdot \nabla_p \chi_{\mathrm{q}}(\theta, X)\big)\, d\theta\right) \partial_E F(t, H_0(X)) \\
&\quad - \left(\int_{\mathbb{Y}} P\big(\nabla_x V_{\mathrm{q}}(\theta, x) \cdot \nabla_p H_0(X)\, \chi_{\mathrm{q}}(\theta, X)\big)\, d\theta\right) \partial_{EE}^2 F(t, H_0(X)).
\end{aligned}$$

Thanks to the coarea formula (A.5), we deduce that $F(t, E)$ verifies the following drift-diffusion equation

$$\partial_t\big(h_0(E)F(t, E)\big) = h_0(E)a(E)\partial_E F(t, E) + h_0(E)b(E)\partial_{EE}^2 F(t, E), \tag{3.4}$$

the coefficients of which are defined by

$$\begin{cases}
a(E) = \Pi\left(\displaystyle\int_{\mathbb{Y}} \nabla_x V_{\mathrm{q}}(\theta, x) \cdot \nabla_p \chi_{\mathrm{q}}(\theta, X)\, d\theta\right)(E), \\
b(E) = \Pi\left(\displaystyle\int_{\mathbb{Y}} \nabla_x V_{\mathrm{q}}(\theta, x) \cdot \nabla_p H_0(X)\, \chi_{\mathrm{q}}(\theta, X)\, d\theta\right)(E).
\end{cases}$$

For further purposes, it is also convenient to introduce χ_{q}^*, a solution of the adjoint profile equation

$$\mathcal{T}^* \chi_{\mathrm{q}}^* = \nabla_x V_{\mathrm{q}}(\theta, x) \cdot \nabla_p H_0(X).$$

This function is precisely defined in Corollary 4.4 below. Let us set

$$\begin{cases} a^*(E) = \Pi\left(\int_{\mathbb{Y}} \nabla_x V_{\mathrm{q}}(\theta, x) \cdot \nabla_p \chi^*_{\mathrm{q}}(\theta, X)\, d\theta\right)(E) \\ b^\star(E) = \Pi\left(\int_{\mathbb{Y}} \nabla_x V_{\mathrm{q}}(\theta, x) \cdot \nabla_p H_0(X)\; \chi^*_{\mathrm{q}}(\theta, X)\, d\theta\right)(E). \end{cases} \tag{3.5}$$

The following claim will make the connection with (1.7) clear.

Lemma 3.1. *The following relations hold true:*

$$h_0(E)b^*(E) = h_0(E)b(E), \quad h_0(E)a^\star(E) = h_0(E)a(E) = \frac{d}{dE}\left(h_0(E)b^*(E)\right).$$

These relations are consequences of the coarea formula; detailed computations are presented in Appendix C.1. Therefore, from (3.4), we are led to (1.7):

$$\partial_t(h_0 F) = \partial_E(h_0 b)\partial_E F(t, E) + h_0(E)b(E)\partial^2_{EE} F(t, E) = \partial_E(h_0 b \partial_E F).$$

We are now left with the task of making this formal guess rigorous. To this end, we need some technical assumptions on the perturbation V.

Hypothesis 3.1. *We assume that*

(i) the quasiperiodic potential $V(t, x) = V_{\mathrm{q}}(\omega t, x)$ posesses the regularity $V_{\mathrm{q}} \in C^2_b(\mathbb{Y} \times \mathbb{R}^d)$, where V_{q} is $\mathbb{Y}-$periodic with respect to the first variable.

(ii) There exists some $\beta \geq 0$ such that

$$\sup_{\theta \in \mathbb{Y}} \int_{\mathbb{R}^{2d}} \frac{|\nabla_x V_{\mathrm{q}}(\theta, x) \cdot \nabla_p H_0(X)|^2}{w(X)^\beta}\, dX < \infty,$$

where $w(X) = (1 + H_0(X)^2)^{1/2}$.

Remark 3.1. *Considering the harmonic Hamiltonian, we get $\nabla_x V \cdot \nabla_p H_{\mathrm{harm}}(X) p \cdot \nabla_x V$ which clearly does not belong to $L^2(\mathbb{R}^{2d})$. However, Hypothesis 3.1-(ii) holds for any $\beta > d + 1$. Thus, the ε order equation* (3.2) *makes sense in a reasonable functional space since the right-hand side belongs to the weighted space $L^2(\mathbb{R}^{2d}, w(X)^{-\beta} dX)$.*

We are now ready to give the statement of our main result.

Theorem 3.2. *Let $f_0^\varepsilon \geq 0$ be the initial data for* (1.2). *We suppose that f_0^ε is bounded in $L^2(\mathbb{R}^{2d})$. We suppose that Hypothesis (1.1), (1.2), (1.3) and 3.1 are satisfied. Then, $f^\varepsilon = Pf^\varepsilon + \varepsilon g_\varepsilon$ where g_ε is bounded in $L^2((0, T) \times \mathbb{R}^{2d})$ and, up to a subsequence, $Pf^\varepsilon(t, X)$ converges in $C^0([0, T]; L^2(\mathbb{R}^{2d}) - weak)$*

to $F(t, H_0(X))$, where $F : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ satisfies the diffusion equation (1.7) *weakly in $L^2(\mathbb{R})$, with the initial data $F(t = 0, E)$ given by the weak limit of $\Pi f_0^\varepsilon(E)$ in $L^2(\mathbb{R}, h_0(E)\, dE)$.*

Remark 3.2. *We point out that assuming $\gamma > 0$ is crucial in our analysis since the operator Q plays the role of a dissipation which allows to avoid all resonance phenomena. The explicit computations presented in Appendix E.1 may shed some light on this aspect. Without such a relaxation, the mathematical analysis becomes very delicate and certainly does not lead to a diffusion process. We refer in particular to Ref. (2, 20) where it is shown that the homogenization of a kinetic equation with highly oscillatory force fields leads to an effective equation involving memory effects. These results are in the spirit of those concerning the homogenization of transport equations with transverse oscillations[1,4,32] as initiated by Ref. (38). In the present approach, we avoid these effects thanks to the presence of a dissipation operator.*

4. PROFILE EQUATIONS

This section is devoted to the analysis of the profile equation $\mathcal{T} f_q = h_q$. We denote by $L^2_\#(\mathbb{Y} \times \mathbb{R}^{2d})$ the class of functions $f_q : \mathbb{R}^r \times \mathbb{R}^{2d} \to \mathbb{R}$ which are $\mathbb{Y}-$periodic with respect to the first variable and such that

$$\int_{\mathbb{Y}\times\mathbb{R}^{2d}} |f_q(\theta, X)|^2\, d\theta\, dX < \infty.$$

We also introduce

$$H_\# = \left\{ f_q \in L^2_\#(\mathbb{Y} \times \mathbb{R}^{2d}),\ \mathcal{T} f_q \in L^2_\#(\mathbb{Y} \times \mathbb{R}^{2d}) \right\}.$$

4.1. General Setting

Proposition 4.1. *Let $h_q \in L^2_\#(\mathbb{Y} \times \mathbb{R}^{2d})$. We suppose that h_q is either purely periodic or has finitely many harmonics, which means that either $r = 1$, or, when $r \geq 2$,*

$$h_q(\theta, x) = \sum_{k\in\mathbb{Z}^r,\ |k|\leq K} \widehat{h_q}(k, x)\, \exp(ik\cdot\theta), \tag{4.1}$$

for some finite integer K. Then, the problem $\mathcal{T} f_q = h_q$ has a solution $f_q \in H_\#$ iff h_q satisfies the compatibility condition

$$\int_{\mathbb{Y}} P h_q(\theta, X)\, d\theta = 0. \tag{4.2}$$

The solution is unique when imposing the additional constraint $\int_{\mathbb{Y}} Pf_{\mathrm{q}}(\theta, X)\, d\theta = 0$. *This uniquely defined solution depends continuously on* h_{q}*: there exists* $C > 0$ *such that*

$$\|f_{\mathrm{q}}\|_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})} \leq C\ \|h_{\mathrm{q}}\|_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})}.$$

Other solutions differ from f_{q} *by a function* $\varphi(H_0(X))$.

Proof. The arguments are inspired from Ref. (21), but specific difficulties appear, since in particular the operators $\omega \cdot \nabla_\theta$ and $\{H_0, \cdot\} - Q$ act on independent variables. As it will become clear in the proof, the restriction contained in (4.1) is related to small denominator difficulties when solving the profile equations. These difficulties disappear in the purely periodic case. The proof splits as follows.

Uniqueness

For any $f_{\mathrm{q}} \in H_\#$, we observe that

$$\int_{\mathbb{Y}\times\mathbb{R}^{2d}} \omega \cdot \nabla_\theta f_{\mathrm{q}}\ f_{\mathrm{q}}\, d\theta\, dX = 0, \qquad \int_{\mathbb{Y}\times\mathbb{R}^{2d}} \{H_0, f_{\mathrm{q}}\}\ f_{\mathrm{q}}\, d\theta\, dX = 0.$$

Let $f_{\mathrm{q}} \in H_\#$ be a solution of $\mathcal{T} f_{\mathrm{q}} = 0$. Multiplying by f_{q} and integrating yields

$$-\gamma \int_{\mathbb{Y}\times\mathbb{R}^{2d}} Q(f_{\mathrm{q}}) f_{\mathrm{q}}\, d\theta\, dX = 0 = \gamma\, \|f_{\mathrm{q}} - Pf_{\mathrm{q}}\|^2_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})}$$

thanks to Corollary 2.2. We deduce that $f_{\mathrm{q}}(\theta, X) = Pf_{\mathrm{q}}(\theta, X)$ depends on X only through the energy. Then, we apply the operator P to the equation. We get

$$\omega \cdot \nabla_\theta P f_{\mathrm{q}} = 0$$

thanks to Lemma 2.1-(ii) and (v). Accordingly, the Fourier coefficients of Pf_{q} verify

$$\omega \cdot k\ \widehat{Pf_{\mathrm{q}}}(k, X) = 0.$$

Since the components of the frequency vector ω are assumed rationally independent, we deduce that $\widehat{Pf_{\mathrm{q}}}(k, X) = 0$ for any $k \neq 0$, and thus this implies that $Pf_{\mathrm{q}}(\theta, X)$ does not depend on the variable $\theta \in \mathbb{Y}$. We proved that $f_{\mathrm{q}} \in L^2(\mathbb{Y} \times \mathbb{R}^{2d})$ verifies $\mathcal{T} f_{\mathrm{q}} = 0$ iff $f_{\mathrm{q}}(\theta, X) = F(H_0(X))$, for some F such that $\int_{\mathbb{R}^{2d}} |F(H_0(X))|^2\, dX < \infty$. In particular, if we impose that $\int_{\mathbb{Y}} Pf_{\mathrm{q}}\, d\theta = 0$, this implies that $f_{\mathrm{q}} = 0$, proving the uniqueness result.

Existence

Applying the projector P to the equation $\mathcal{T} f_{\mathrm{q}} = h_{\mathrm{q}}$ and integrating over $\mathbb{Y}$, we realize that (4.2) is a necessary condition for having a solution. From now on, we thus assume that (4.2) holds true and we prove that it is also a sufficient condition. Let us temporarily assume that, for any $\lambda > 0$, there exists $f_{\mathrm{q}}^{(\lambda)} \in H_{\#}$ verifying

$$\lambda f_{\mathrm{q}}^{(\lambda)} + \mathcal{T} f_{\mathrm{q}}^{(\lambda)} = h_{\mathrm{q}}. \tag{4.3}$$

We wish to prove the existence part of Proposition 4.1 by passing to the limit $\lambda \to 0$. This is completely obvious once we know that the sequence $\big(f_{\mathrm{q}}^{(\lambda)}\big)_{\lambda>0}$ remains bounded in $L^2(\mathbb{Y} \times \mathbb{R}^{2d})$.

Suppose that there exists a subsequence, say $\{\lambda^{(n)}, n \in \mathbb{N}\}$ such that $\lim_{n\to\infty} \lambda^{(n)} = 0$ and

$$N^{(n)} = \big\| f_{\mathrm{q}}^{(\lambda_n)} \big\|_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})} \longrightarrow_{n\to\infty} +\infty.$$

We set $F_{\mathrm{q}}^{(n)} = f_{\mathrm{q}}^{(\lambda_n)}/N^{(n)}$. Without loss of generality, we can assume that $F_{\mathrm{q}}^{(n)} \rightharpoonup F_{\mathrm{q}}$ weakly in $L^2(\mathbb{Y} \times \mathbb{R}^{2d})$ as $n \to \infty$. We have

$$\lambda^{(n)} F_{\mathrm{q}}^{(n)} + \mathcal{T} F_{\mathrm{q}}^{(n)} = \frac{h_{\mathrm{q}}}{N^{(n)}}.$$

Hence, multiplying by $F_{\mathrm{q}}^{(n)}$ leads to

$$\gamma \, \big\| F_{\mathrm{q}}^{(n)} - P F_{\mathrm{q}}^{(n)} \big\|^2_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})} \le \int_{\mathbb{Y}\times\mathbb{R}^{2d}} \frac{h_{\mathrm{q}}}{N^{(n)}} \, F_{\mathrm{q}}^{(n)} \, d\theta \, dX \le \frac{\|h_{\mathrm{q}}\|_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})}}{N^{(n)}}.$$

We deduce that

$$\big\| F_{\mathrm{q}}^{(n)} - P F_{\mathrm{q}}^{(n)} \big\|^2_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})} \longrightarrow_{n\to\infty} 0. \tag{4.4}$$

Accordingly, $F_{\mathrm{q}}^{(n)} = P F_{\mathrm{q}}^{(n)} + (F_{\mathrm{q}}^{(n)} - P F_{\mathrm{q}}^{(n)}) \rightharpoonup F_{\mathrm{q}} = P F_{\mathrm{q}}$ as $n \to \infty$. Now, we apply the projection operator and we get

$$\lambda^{(n)} P F_{\mathrm{q}}^{(n)} + \omega \cdot \nabla_\theta P F_{\mathrm{q}}^{(n)} = \frac{P h_{\mathrm{q}}}{N^{(n)}}. \tag{4.5}$$

Integrating with respect to θ, we obtain for any $n \in \mathbb{N}$

$$\int_{\mathbb{Y}} P F_{\mathrm{q}}^{(n)}(\theta, X) \, d\theta = 0,$$

as a consequence of (4.2). Besides, passing to the limit in (4.5) yields

$$\omega \cdot \nabla_\theta P F_{\mathrm{q}}^{(n)} \underset{n\to\infty}{\longrightarrow} \omega \cdot \nabla_\theta P F_{\mathrm{q}} = 0 \quad \text{strongly in } L^2(\mathbb{Y} \times \mathbb{R}^{2d}).$$

Hence the limit is nothing but $F_{\mathrm{q}} = 0$. We will obtain a contradiction by showing that $F_{\mathrm{q}}^{(n)}$ converges strongly.

Let us consider the Fourier series associated with $P F_{\mathrm{q}}^{(n)}$

$$P F_{\mathrm{q}}^{(n)}(\theta, X) = \sum_{k\in\mathbb{Z}^r} \widehat{P F}_{\mathrm{q}}^{(n)}(k, X) e^{2i\pi k\cdot\theta}.$$

We have already remarked that the first Fourier coefficient vanishes

$$\widehat{PF}_{\mathrm{q}}^{(n)}(0, X) = \int_{\mathbb{Y}} PF_{\mathrm{q}}^{(n)}(\theta, X)\, d\theta = 0.$$

Therefore, the Plancherel theorem gives

$$\begin{aligned} \| PF_{\mathrm{q}}^{(n)} \|^2_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})} &= \sum_{k\in\mathbb{Z}^r\setminus\{0\}} \big|\widehat{PF}_{\mathrm{q}}^{(n)}(k, X)\big|^2 \\ &= \sum_{k\in\mathbb{Z}^r\setminus\{0\}} \frac{1}{|\omega\cdot k|^2}\, |\omega\cdot k|^2 |\widehat{PF}_{\mathrm{q}}^{(n)}(k, X)|^2 \\ &= \sum_{k\in\mathbb{Z}^r\setminus\{0\}} \frac{1}{4\pi^2|\omega\cdot k|^2}\, |\omega\cdot\nabla_\theta \widehat{PF}_{\mathrm{q}}^{(n)}(k, X)|^2. \end{aligned}$$

When $r \geq 2$, we use the assumption that the data h_{q} has finitely many harmonics. By (4.5), $PF_{\mathrm{q}}^{(n)}$ shares the same property, with the same truncation index K and we are thus led to

$$\| PF_{\mathrm{q}}^{(n)} \|^2_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})} \leq \sup_{k\in\mathbb{Z}^r\setminus\{0\},\ |k|\leq K} \left(\frac{1}{4\pi^2|\omega\cdot k|^2}\right) \| \omega\cdot\nabla_\theta PF_{\mathrm{q}}^{(n)} \|^2_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})} \underset{n\to\infty}{\longrightarrow} 0.$$

When $r = 1$ the conclusion is immediate since we get $\| PF_{\mathrm{q}}^{(n)} \|^2_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})} \leq \| \partial_\theta PF_{\mathrm{q}}^{(n)} \|^2_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})}$.

It remains to justify the existence of $F^{(\lambda)}$. This is obtained by a Banach fixed point argument. Indeed, consider the operator $\Phi^{(\lambda)}$, which to a function $\phi \in L^2_\#(\mathbb{Y}\times\mathbb{R}^{2d})$ associates the solution $\psi^{(\lambda)} = \Phi^{(\lambda)}(\phi) \in L^2_\#(\mathbb{Y}\times\mathbb{R}^{2d})$ to the transport equation

$$\lambda\psi^{(\lambda)}(\theta, X) + \omega\cdot\nabla_\theta\psi^{(\lambda)} + \big\{H_0, \psi^{(\lambda)}\big\} + \gamma\,\psi^{(\lambda)} = \gamma\, P\phi + h_{\mathrm{q}}(\theta, X).$$

We prove that $\Phi^{(\lambda)}$ is a contraction over $L^2_\#(\mathbb{Y}\times\mathbb{R}^{2d})$. Since (4.3) also reads $f_{\mathrm{q}}^{(\lambda)} = \Phi^{(\lambda)}(f_{\mathrm{q}}^{(\lambda)})$, this clearly implies the existence and uniqueness of $f_{\mathrm{q}}^{(\lambda)}$, the solution to (4.3). Now, to prove the contraction property of $\Phi^{(\lambda)}$, we take two functions ϕ and $\widetilde{\phi}$, with the associated $\psi^{(\lambda)} = \Phi^{(\lambda)}(\phi)$ and $\widetilde{\psi}^{(\lambda)} = \Phi^{(\lambda)}(\widetilde{\phi})$. We readily obtain the following energy estimate

$$\begin{aligned} (\lambda+\gamma)\big\|\psi^{(\lambda)} - \widetilde{\psi}^{(\lambda)}\big\|^2_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})} &\leq \gamma\,\big|\big\langle P\phi - P\widetilde{\phi}, \psi^{(\lambda)} - \widetilde{\psi}^{(\lambda)}\big\rangle_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})}\big| \\ &\leq \gamma\,\big\|\phi - \widetilde{\phi}\big\|_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})}\big\|\psi^{(\lambda)} - \widetilde{\psi}^{(\lambda)}\big\|_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})}. \end{aligned}$$

The second estimate uses the Cauchy-Schwarz inequality together with the continuity of P over $L^2(\mathbb{Y}\times\mathbb{R}^{2d})$ (see Lemma 2.1). As a consequence, we have

$$\big\|\psi^{(\lambda)} - \widetilde{\psi}^{(\lambda)}\big\|_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})} \leq \frac{\gamma}{\gamma+\lambda}\,\big\|\phi - \widetilde{\phi}\big\|_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})}.$$

This is the claimed contraction property.

This ends the proof of Proposition 4.1. The continuity estimate follows from the closed graph theorem, once we have remarked that the set of functions verifying the compatibility condition is a closed subspace of $L^2_\#(\mathbb{Y} \times \mathbb{R}^{2d})$. ∎

The distinction between the purely periodic case and the genuinely quasiperiodic case is due to small denominator difficulties: while the transport operator ∂_θ is (essentially) invertible over $L^2(d\theta)$ in one dimension, the inverse of the transport operator $\omega \cdot \nabla_\theta$ ceases to be bounded in reasonable spaces when the angular variable θ belongs to the $r > 1$ dimensional torus. This appears clearly when we try to deduce the behavior of $PF^{(n)}$ from informations on $\omega \cdot \nabla_\theta PF^{(n)}$. In the periodic case the required estimate is actually nothing but the classical Poincaré-Wirtinger estimate for periodic functions on (0, 1). When $r \geq 2$, the quantity $|\omega \cdot k|^2$ is never zero when $k \neq 0$, due to the rational independence of the components of ω. Nevertheless, small denominators might appear, corresponding to cases where $\omega \cdot k$ is small but nonzero. This typically happens for large values of $|k|$. This is the reason why we assume, in the case $r \geq 2$, that h_{q} has finitely many harmonics. Another (classical) way to analyze this difficulty consists in saying that the Fredholm alternative does not apply to the transport operator $\omega \cdot \nabla_\theta$; its range is not closed in general. The difficulty can also be illustrated by imposing some diophantine condition on ω (which is therefore satisfied for almost all ω). Some slight adaptations of the previous proof then lead to the following claim

Proposition 4.2. *Let ω satisfy the following diophantine condition: for any $k \in \mathbb{Z}^r$,*

$$|\omega \cdot k| \geq \frac{C_\gamma}{|k|^\gamma},$$

holds for some $\gamma > 0$ and $C_\gamma > 0$. Let $h_{\mathrm{q}} \in L^2_\#(\mathbb{Y} \times \mathbb{R}^{2d})$ satisfy

$$\|Ph_{\mathrm{q}}\|^2_{H^\gamma_\#(\mathbb{Y};L^2(\mathbb{R}^{2d}))} := \sum_{k \in \mathbb{Z}^r} |k|^{2\gamma} \|\widehat{Ph_{\mathrm{q}}}(k,\cdot)\|^2_{L^2(\mathbb{R}^{2d})} < \infty.$$

Then, the problem $\mathcal{T} f_{\mathrm{q}} = h_{\mathrm{q}}$ has a solution $f_{\mathrm{q}} \in H_\#$ iff h_{q} satisfies the compatibility condition (4.2). *The solution is unique when imposing the additional constraint $\int_{\mathbb{Y}} Pf_{\mathrm{q}}(\theta, X)\, d\theta = 0$. This uniquely defined solution depends continuously on h_{q} in the sense that*

$$\|(I-P)f_{\mathrm{q}}\|_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})} \leq C\, \|h_{\mathrm{q}}\|_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})}, \quad \|Pf_{\mathrm{q}}\|_{L^2(\mathbb{Y}\times\mathbb{R}^{2d})} \leq C\, \|h_{\mathrm{q}}\|_{H^\gamma(\mathbb{Y};L^2(\mathbb{R}^{2d}))}.$$

Other solutions differ from f_{q} by a function $\varphi(H_0(X))$.

In the course of the formal derivation, we have seen that we actually have to consider data belonging to some weighted space:

$$h_{\mathrm{q}} : \mathbb{Y} \times \mathbb{R}^{2d} \to \mathbb{R}, \quad \mathbb{Y} - \text{periodic}, \quad \int_{\mathbb{Y}\times\mathbb{R}^{2d}} |h_{\mathrm{q}}(\theta, X)|^2\, w(X)^\alpha\, dX\, d\theta < \infty$$

for some real α. The profile equation in such a weighted space is easily reduced to the simpler L^2 framework. Indeed, define $\widetilde{h}_{\mathrm{q}}(\theta, X) = h_{\mathrm{q}}(\theta, X) w(X)^{\alpha/2}$. Then, $\widetilde{h}_{\mathrm{q}}$ belongs to $L^2_{\#}(\mathbb{Y} \times \mathbb{R}^{2d})$. Hence, we solve $\mathcal{T}\widetilde{f}_{\mathrm{q}} = \widetilde{h}_{\mathrm{q}}$ with $\widetilde{f}_{\mathrm{q}} \in H_{\#}$, $\int_{\mathbb{Y}} P\widetilde{f}_{\mathrm{q}}\, d\theta = 0$ and we set $f_{\mathrm{q}}(\theta, X) = \widetilde{f}_{\mathrm{q}}(\theta, X) w(X)^{-\alpha/2}$. f_{q} satisfies

$$\int_{\mathbb{Y}\times\mathbb{R}^{2d}} |f_{\mathrm{q}}(\theta, X)|^2\, w(X)^{\alpha}\, dX\, d\theta < \infty, \quad \mathcal{T} f_{\mathrm{q}} = h_{\mathrm{q}}, \quad \int_{\mathbb{Y}} P f_{\mathrm{q}}\, d\theta = 0$$

since multiplication by a (smooth enough) function of $H_0(X)$ commutes with the operator $\mathcal{T}$. Clearly, similar conclusions hold for the adjoint operator $\mathcal{T}^*$, which shows that the results can easily be extended to the weighted space framework.

Let us now turn to the very particular case we are interested in.

4.2. Solution of the Profile Equation (3.2)

The computation of the effective coefficients relies on the resolution of the profile equation with data $\nabla_x V_{\mathrm{q}}(\theta, x) \cdot \nabla_p H_0(x, p)$. The compatibility condition (4.2) is satisfied in a strong way since we actually have

$$P(\nabla_x V_{\mathrm{q}} \cdot \nabla_p H_0) = P\{V_{\mathrm{q}}, H_0\} = 0.$$

This allows us to derive a more explicit expression for the solution χ_{q} (resp. χ^*_{q}) of the profile equation $\mathcal{T}\chi_{\mathrm{q}} = \nabla_x V_{\mathrm{q}} \cdot \nabla_p H_0$ (resp. $\mathcal{T}^*\chi^*_{\mathrm{q}} = \nabla_x V_{\mathrm{q}} \cdot \nabla_p H_0$).

Indeed, let us consider the profile equation $\mathcal{T} f_{\mathrm{q}} = h_{\mathrm{q}}$ under the condition $Ph_{\mathrm{q}} = 0$. (Similar computations can be performed for the adjoint equation.) Then, applying the operator P to the equation yields $\omega \cdot \nabla_\theta P f_{\mathrm{q}} = Ph_{\mathrm{q}} = 0$ which implies that Pf_{q} does not depend on θ. Requiring $\int_{\mathbb{Y}} Pf_{\mathrm{q}}\, d\theta = 0$ gives $Pf_{\mathrm{q}} = 0$. Therefore, we are led to solve

$$\begin{cases} \omega \cdot \nabla_\theta f_{\mathrm{q}} + \{H_0, f_{\mathrm{q}}\} + \gamma f_{\mathrm{q}} = h_{\mathrm{q}}, \\ Pf_{\mathrm{q}} = 0. \end{cases}$$

Let us introduce the characteristics $\Theta \in \mathbb{R}^r$, $\overline{X} \in \mathbb{R}^{2d}$, the solutions of the ODEs system

$$\begin{cases} \dfrac{d}{ds}\Theta(s) = \omega, & \dfrac{d}{ds}\overline{X}(s) = \big(\nabla_p H_0(\overline{X}(s)), -\nabla_x H_0(\overline{X}(s))\big), \\ \Theta(0) = \theta, & \overline{X}(0) = (x, p). \end{cases}$$

Note in particular that $\Theta(s) = \theta + s\omega$. Hence, we get

$$\frac{d}{ds}\big(e^{\gamma s}\, f_{\mathrm{q}}(\Theta(s), \overline{X}(s))\big) = e^{\gamma s}\, h_{\mathrm{q}}(\Theta(s), \overline{X}(s)).$$

Integration with respect to s yields the following statement:

Lemma 4.3. *Let $h_q \in L^2(\mathbb{Y} \times \mathbb{R}^{2d})$ be such that $Ph_q = 0$. Then the solution $f_q \in H_\#$ of $\mathcal{T}h_q = h_q$ with $Pf_q = 0$ is given by*

$$f_q(\theta, x, p) = \int_{-\infty}^{0} e^{\gamma s}\, h_q(\Theta(s), \overline{X}(s))\, ds. \tag{4.6}$$

Accordingly, if h_q lies in $C^0(\mathbb{Y}; L^2(\mathbb{R}^{2d}))$, then, f_q lies in the same space. If, furthermore $\nabla_X h_q$ lies in $C^0(\mathbb{Y}; L^2_{\text{loc}}(\mathbb{R}^{2d}))$, then, f_q also satisfies this property.

There only remains to discuss the regularity statement, which follows from a direct application of Lebesgue's dominated convergence theorem. Similarly, we can differentiate (4.6) with respect to X and conclude thanks to Hypothesis (1.3). Let us now state the precise result which will be used in the sequel:

Corollary 4.4. *Assume Hypothesis 1.1, 1.2, 1.3, 3.1. Then, there exists a unique function $\chi_q^* : \mathbb{Y} \times \mathbb{R}^{2d} \to \mathbb{R}$ such that*

$$\int_{\mathbb{Y}\times\mathbb{R}^{2d}} \left|\chi_q^*(\theta, X)\right|^2 \frac{dX\, d\theta}{w(X)^\beta} < \infty, \quad \mathcal{T}^*\chi_q^* = \nabla_x V_q \cdot \nabla_p H_0, \quad \int_{\mathbb{Y}} P\chi_q^*\, d\theta = 0.$$

It is defined by the formula

$$\chi_q^*(\theta, x, p) = \int_0^\infty e^{-\gamma s}\, \nabla_x V_q \cdot \nabla_p H_0(\theta + s\omega, \overline{X}(s; x, p))\, ds.$$

Furthermore, for any $0 < R < \infty$, χ_q^* and $\nabla_X \chi_q^*$ belong to $C^0(\mathbb{Y}; L^2(B(0, R)))$, where $B(0, R) = \{X \in \mathbb{R}^{2d}, |X| \leq R\}$, and $P\chi_q^* = 0$.

Remark 4.1. *The role of Hypothesis 1.3 is to guarantee that χ_q^* possesses enough regularity to justify some algebraic manipulations below. If, instead of Hypothesis 1.3, we assume the weaker hypothesis $H_0 \in W^{2,\infty}(\mathbb{R}^{2d})$, we readily obtain the following estimate: $|\nabla_{x,p}\overline{X}(s)| \leq e^{Cs}\,(1 + |(x, p)|)$ for some $C > 0$. Then, all our results will remain true provided that we consider large enough values of the parameter γ (which should be $> C$). However, this looks too strong a restriction from a physical viewpoint because usually, relaxation rates are rather weak.*

5. PROOF OF THEOREM 3.2

5.1. A Priori Estimates

We obtain the basic uniform estimate by multiplying (1.2) by f^ε and performing some integration by parts. Since the transport terms are antisymmetric, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^{2d}} |f^\varepsilon|^2\, dX = \frac{\gamma}{\varepsilon^2}\int_{\mathbb{R}^{2d}} Q(f^\varepsilon) f^\varepsilon\, dX = -\frac{\gamma}{\varepsilon^2}\int_{\mathbb{R}^{2d}} |Pf^\varepsilon - f^\varepsilon|^2\, dX \leq 0,$$

thanks to Corollary 2.2. Hence, we deduce the following claim.

Proposition 5.1. *Suppose that the initial data f_0^ε is bounded in $L^2(\mathbb{R}^{2d})$. Then,*

(i) $\left(f^\varepsilon\right)_{\varepsilon>0}$ *is bounded in* $L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^{2d}))$,
(ii) $\left(g^\varepsilon = \frac{1}{\varepsilon}(f^\varepsilon - Pf^\varepsilon)\right)_{\varepsilon>0}$ *is bounded in* $L^2(\mathbb{R}^+ \times \mathbb{R}^{2d})$.

Remark 5.1. *For any convex function* $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$, *we have*

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^{2d}} \Psi(f^\varepsilon)\,dX &= \gamma \int_{\mathbb{R}^{2d}} Q(f^\varepsilon)\Psi'(f^\varepsilon)\,dX \\ &= -\frac{\gamma}{\varepsilon^2}\int_{\mathbb{R}^{2d}} \left(Pf^\varepsilon - f^\varepsilon\right)\left(\Psi'(Pf^\varepsilon) - \Psi'(f^\varepsilon)\right)dX \le 0\end{aligned}.$$

In particular, this provides uniform estimates of f^ε in any $L^p(\mathbb{R}^{2d})$ space, $1 \le p \le \infty$. However, these estimates will not be needed in the sequel.

5.2. Convergence Proof

A possible proof would consist in solving the successive profile equations (3.1)–(3.3), constructing an approximate solution $f^\varepsilon_{\rm app} = f^{(0)} + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)}$, evaluating the difference $f^\varepsilon - f^\varepsilon_{\rm app}$ and showing that it is $\mathcal{O}(\varepsilon)$. Such an approach is usually very demanding in terms of regularity of the solution and would lead to tedious technicalities. Moreover, the resolution of the profile equation (3.3) can impose more restrictions on the potential $V_{\rm q}$ than those detailed in Proposition 4.1. Here, we adopt another viewpoint, trying to pass to the limit in the equation. To this end, we follow the general homogenization strategy developed e.g. in Ref. (22). It combines double scale convergence tools, as introduced in Ref. (3, 29), combined with a suitable choice of test functions, the so-called "oscillating test functions method".[(18,19,37,38)] First of all, let us give the following double scale convergence statement, which is adapted to the quasi-periodic framework.

Proposition 5.2. *Let f_ε be a bounded sequence in $L^2(\mathbb{R})$. Let $\omega \in \mathbb{R}^r$ the components of which are rationally independent. Then, there exists a subsequence, still labelled by ε, and a function $F_{\rm q} \in L^2_\#(\mathbb{R} \times \mathbb{Y})$ such that for any test function $\psi_{\rm q} \in L^2(\mathbb{R}; C^0_\#(\mathbb{Y}))$,[5] we have*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} f_\varepsilon(t)\,\psi_{\rm q}(t, \omega t/\varepsilon^2)\,dt = \int_{\mathbb{R}}\int_{\mathbb{Y}} F_{\rm q}(t,\theta)\,\psi_{\rm q}(t,\theta)\,d\theta\,dt.$$

The proof follows the arguments of Ref. (3), which are combined to the ergodic condition "ω has rationally independent components", through the use of a variant of the Birkhoff theorem (see Ref. (15)). This is detailed in Appendix D.1. Further

[5] Referring to Ref. (3) Section 5, $L^2(\mathbb{R}; C^0_\#(\mathbb{Y}))$ is the class of functions $\psi_{\rm q} : \mathbb{R} \times \mathbb{R}^r \to \mathbb{R}$ which are measurable and square integrable with respect to the variable $t \in \mathbb{R}$, with values in the Banach space of continuous and $\mathbb{Y}-$periodic functions.

adaptations to sequences of functions with values in a Hilbert space can be readily obtained as in Ref. (21). Therefore, coming back to Proposition 5.1, we have the following compactness property, where $C^0_{c,\#}(\mathbb{R}\times\mathbb{Y}; L^2(\mathbb{R}^{2d}))$ denotes the space of functions $\psi_{\mathrm{q}} : \mathbb{R}\times\mathbb{R}^r\times\mathbb{R}^{2d}\to\mathbb{R}$ which are continuous with respect to $(t,\theta)\in\mathbb{R}\times\mathbb{R}^r$, $\mathbb{Y}-$ periodic with respect to the second variable, with values in $L^2(\mathbb{R}^{2d})$, and such that $\psi_{\mathrm{q}}(t,\theta,X)=0$ when $t\notin K$, for some compact set $K\subset\mathbb{R}$.

Lemma 5.3. *We can suppose, up to the extraction of a subsequence, that* f^ε *converges to* $F_{\mathrm{q}}(t,\theta,X)\in L^2_\#((0,T)\times\mathbb{Y}\times\mathbb{R}^{2d})$ *in the sense that*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}^{2d}} f^\varepsilon(t,X)\,\psi_{\mathrm{q}}(t,\omega t/\varepsilon^2,X)\,dt$$
$$=\int_{\mathbb{R}}\int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} F_{\mathrm{q}}(t,\theta,X)\,\psi_{\mathrm{q}}(t,\theta,X)\,d\theta\,dX\,dt,$$

holds for any trial function $\psi_{\mathrm{q}}\in C^0_{c,\#}(\mathbb{R}\times\mathbb{Y}; L^2(\mathbb{R}^{2d}))$. *Furthermore,* f^ε *converges weakly in* $L^2((0,T)\times\mathbb{R}^{2d})$ *to* $f(t,X)=\int_{\mathbb{Y}}F_{\mathrm{q}}(t,\theta,X)\,d\theta$.

Let us multiply (1.2) by $\psi_{\mathrm{q}}(t,\omega t/\varepsilon^2,X)$, where ψ_{q} is a C^∞ function of its arguments and is $\mathbb{Y}-$periodic with respect to the second variable. Integrations by parts yield

$$\begin{aligned}
&\varepsilon\frac{d}{dt}\int_{\mathbb{R}^{2d}} f^\varepsilon(t,X)\,\psi_{\mathrm{q}}(t,\omega t/\varepsilon^2,X)\,dX-\varepsilon\int_{\mathbb{R}^{2d}} f^\varepsilon(t,X)\,\partial_t\psi_{\mathrm{q}}(t,\omega t/\varepsilon^2,X)\,dX\\
&-\frac{1}{\varepsilon}\int_{\mathbb{R}^{2d}} f^\varepsilon(t,X)\;\omega\cdot\nabla_\theta\psi_{\mathrm{q}}(t,\omega t/\varepsilon^2,X)\,dX\\
&-\frac{1}{\varepsilon}\int_{\mathbb{R}^{2d}} f^\varepsilon(t,X)\,\big\{H_0,\psi_{\mathrm{q}}\big\}(t,\omega t/\varepsilon^2,X)\,dX\\
&+\int_{\mathbb{R}^{2d}} f^\varepsilon(t,X)\,\nabla_x V_{\mathrm{q}}(\omega t/\varepsilon^2,x)\cdot\nabla_p\psi_{\mathrm{q}}(t,\omega t/\varepsilon^2,X)\,dX\\
&-\frac{1}{\varepsilon}\int_{\mathbb{R}^{2d}} f^\varepsilon(t,X)\,\gamma\,Q(\psi_{\mathrm{q}})(t,\omega t/\varepsilon^2,X)\,dX=0
\end{aligned}\tag{5.1}$$

since $Q^*=Q$.

Hence, we first conclude that

$$\lim_{\varepsilon\to0}\int_0^\infty\int_{\mathbb{R}^{2d}} f^\varepsilon(t,X)\big[\omega\cdot\nabla_\theta\psi_{\mathrm{q}}+\{H_0,\psi_{\mathrm{q}}\}+\gamma Q(\psi_{\mathrm{q}})\big](t,\omega t/\varepsilon^2,X)\,dX\,dt=0$$
$$=\int_0^\infty\int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} F_{\mathrm{q}}(t,\theta,X)\,\big[\omega\cdot\nabla_\theta\psi_{\mathrm{q}}+\{H_0,\psi_{\mathrm{q}}\}+\gamma Q(\psi_{\mathrm{q}})\big](t,\theta,X)\,d\theta\,dX\,dt.$$

It implies that the double scale limit F_{q} does not depend on θ and is only a function of the energy; we denote $F_{\mathrm{q}}(t,\theta,X)=F(t,H_0(X))=f(t,X)$.

Next, we remark that for any function only depending on the energy, the most singular term in (5.1) vanishes. Accordingly, let us choose $\psi_{\rm q}(t,\theta,X) = \varphi(H_0(X)) + \varepsilon\phi_{\rm q}(t,\theta,X)$, with $\varphi \in C_c^\infty(\mathbb{R})$, as a test function. We get

$$\lim_{\varepsilon\to 0}\left\{\int_0^\infty\int_{\mathbb{R}^{2d}} f^\varepsilon(t,X)\big[\omega\cdot\nabla_\theta\phi_{\rm q} + \{H_0,\phi_{\rm q}\} + \gamma\,Q(\phi_{\rm q})\big](t,\omega t/\varepsilon^2,X)\,dX\,dt \right.$$
$$\left. - \int_0^\infty\int_{\mathbb{R}^{2d}} f^\varepsilon(t,X)\,\nabla_x V_{\rm q}(\omega t/\varepsilon^2,X)\cdot\nabla_p\big(\varphi(H_0(X))\big)\,dX\,dt\right\} = 0$$
$$= \int_0^\infty\int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} F_{\rm q}(t,\theta,X)\,[\omega\cdot\nabla_\theta\phi_{\rm q} + \{H_0,\phi_{\rm q}\} + \gamma\,Q(\phi_{\rm q})](t,\theta,X)\,d\theta\,dX\,dt$$
$$- \int_0^\infty\int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} F_{\rm q}(t,\theta,X)\nabla_x V_{\rm q}(\theta,X)\cdot\nabla_p H_0(X)\,\partial_E\varphi(H_0(X))\,d\theta\,dX\,dt$$
$$= -\int_0^\infty\int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} F_{\rm q}(t,\theta,X)\nabla_x V_{\rm q}(\theta,X)\cdot\nabla_p H_0(X)\,\partial_E\varphi(H_0(X))\,d\theta\,dX\,dt = 0.$$

Eventually, we choose $\phi_{\rm q}$ depending on φ in such a way that the order $\mathcal{O}(1)$ term in (5.1) also vanishes. This is indeed possible by choosing $\phi_{\rm q}$ a solution of the (adjoint) profile equation

$$\omega\cdot\nabla_\theta\phi_{\rm q} + \{H_0,\phi_{\rm q}\} + \gamma\,Q(\phi_{\rm q}) = -\mathcal{T}^*\phi_{\rm q} = \nabla_x V_{\rm q}(\theta,X)\cdot\nabla_p H_0(X)\,\partial_E\varphi(H_0(X)).$$

Precisely, we set

$$\phi_{\rm q}(\theta,X) = -\chi_{\rm q}^*(\theta,X)\,\partial_E\varphi(H_0(X)).$$

with $\chi_{\rm q}^*$ defined in Corollary 4.4. Note that by the regularity properties in Corollary 4.4, $\phi_{\rm q}(\theta,X)$ and $\nabla_p\phi_{\rm q}(\theta,X) = -\nabla_p\chi_{\rm q}^*\ \partial_E\varphi(H_0(X)) - \chi_{\rm q}^*\ \nabla_p H_0(X)\ \partial^2_{EE}\varphi(H_0(X))$ can indeed be used as "admissible" test functions. It follows that

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}} f^\varepsilon(t,X)\,\big(\varphi(H_0(X)) + \varepsilon\phi_{\rm q}(\omega t/\varepsilon^2,X)\big)\,dX$$
$$+ \int_{\mathbb{R}^{2d}} f^\varepsilon(t,X)\,\nabla_x V_{\rm q}(\omega t/\varepsilon^2,x)\cdot\nabla_p\phi_{\rm q}(\omega t/\varepsilon^2,X)\,dX = 0, \qquad (5.2)$$

holds in $\mathcal{D}'((0,+\infty))$.

Equation (5.2) indicates that

$$\left|\frac{d}{dt}\int_{\mathbb{R}^{2d}} f^\varepsilon(t,X)\,\big(\varphi(H_0(X)) + \varepsilon\phi_{\rm q}(\omega t/\varepsilon^2,X)\big)\,dX\right|$$
$$\le \|f^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^{2d}))}\ \|\nabla_x V_{\rm q}\|_{L^\infty(\mathbb{Y}\times\mathbb{R}^{2d})}\ \|\nabla_p\phi_{\rm q}\|_{L^\infty(\mathbb{Y};L^2(\mathbb{R}^{2d}))},$$

is uniformly bounded with respect to $\varepsilon > 0$, $0 \le t \le T < \infty$, thanks to Proposition 5.1, Hypothesis 3.1, Corollary 4.4 and the fact that φ has a compact support. Hence, for any φ fixed in $C_c^\infty(\mathbb{R})$, the family

$$\left\{ \int_{\mathbb{R}^{2d}} f^\varepsilon(t,X)\big(\varphi(H_0(X)) + \varepsilon\phi_{\mathrm{q}}(\omega t/\varepsilon^2, X)\big)\, dX,\ \varepsilon > 0 \right\}$$

is relatively compact in $C^0([0,T])$, by virtue of the Arzela-Ascoli theorem. But, we also have

$$\begin{aligned} \int_{\mathbb{R}^{2d}} f^\varepsilon(t,X)\,\varphi(H_0(X))\,dX &= \int_{\mathbb{R}^{2d}} Pf^\varepsilon(t,X)\,\varphi(H_0(X))\,dX \\ &= \int_{\mathbb{R}^{2d}} f^\varepsilon(t,X)\big(\varphi(H_0(X)) + \varepsilon\phi_{\mathrm{q}}(\omega t/\varepsilon^2, X)\big)\,dX \\ &\quad - \varepsilon \int_{\mathbb{R}^{2d}} f^\varepsilon(t,X)\,\phi_{\mathrm{q}}(\omega t/\varepsilon^2, X)\,dX \end{aligned}$$

where the last integral is dominated by

$$\|f^\varepsilon\|_{L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^{2d}))}\ \|\chi_{\mathrm{q}}^*\|_{L^\infty(\mathbb{Y};L^2(\{X\in\mathbb{R}^{2d},\ H_0(X)\in \mathrm{supp}\varphi\}))}\ \|\varphi\|_{W^{1,\infty}(\mathbb{R})}.$$

Thus, the family

$$\left\{ \int_{\mathbb{R}^{2d}} Pf^\varepsilon(t,X)\,\varphi(H_0(X))\,dX,\ \varepsilon > 0 \right\}$$

is relatively compact in $C^0([0,T])$. Combining a separability and a diagonal extraction argument, we conclude that we can consider a subsequence, still labelled by ε, such that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^{2d}} Pf^\varepsilon(t,X)\,\varphi(H_0(X))\,dX = \int_{\mathbb{R}^{2d}} F(t, H_0(X))\,\varphi(H_0(X))\,dX$$

uniformly on $[0,T]$, for any φ verifying $\int_{\mathbb{R}^{2d}} |\varphi(H_0(X))|^2\,dX < \infty$.

Furthermore, the limit of the second integral in (5.2) as $\varepsilon \to 0$ reads

$$\begin{aligned} &\int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} F_{\mathrm{q}}(t,\theta,X)\,\nabla_x V_{\mathrm{q}}(\theta,x)\cdot\nabla_p\chi_{\mathrm{q}}^*(\theta,X)\,\partial_E\varphi(H_0(X))\,d\theta\,dX \\ &\quad + \int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} F_{\mathrm{q}}(t,\theta,X)\,\nabla_x V_{\mathrm{q}}(\theta,x)\cdot\nabla_p H_0(X)\,\chi_{\mathrm{q}}^*(\theta,X)\,\partial_{EE}^2\varphi(H_0(X))\,d\theta\,dX \\ &\quad = \int_{\mathbb{R}^{2d}} F(t,H_0(X))\left(\int_{\mathbb{Y}} \nabla_x V_{\mathrm{q}}(\theta,x)\cdot\nabla_p\chi_{\mathrm{q}}^*(\theta,X)\,d\theta\right)\partial_E\varphi(H_0(X))\,dX \\ &\quad + \int_{\mathbb{R}^{2d}} F(t,H_0(X))\left(\int_{\mathbb{Y}} \nabla_x V_{\mathrm{q}}(\theta,x)\cdot\nabla_p H_0(X)\,\chi_{\mathrm{q}}^*(\theta,X)\,d\theta\right)\partial_{EE}^2\varphi(H_0(X))\,dX \\ &\quad = \int_{\mathbb{R}^{2d}} F(t,H_0(X))\big(a^*\partial_E\varphi(H_0(X)) + b^*\partial_{EE}^2\varphi(H_0(X))\big)\,dX, \end{aligned}$$

since we have seen that $F_{\mathrm{q}}(t,\theta,X) = F(t, H_0(X))$. Hence, letting ε tend to 0 in (5.2) yields

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}} F(t, H_0(X))\varphi(H_0(X))\, dX = \int_{\mathbb{R}^{2d}} F(t, H_0(X))\big(a^*\partial_E\varphi(H_0(X)) + b^*\partial^2_{EE}\varphi(H_0(X))\big)\, dX. \quad (5.3)$$

Let us detail some properties of the effective coefficients. ∎

Lemma 5.4. *The coefficients a^* and b^* belong to $L^2_{\mathrm{loc}}(\mathbb{R}, h_0(E)dE)$, and we have $b^*(E) \geq 0$ for almost all $E \in \mathbb{R}$. If furthermore, for any measurable set $A \subset \mathbb{R}$, and $\theta \in \mathbb{Y}$, we have*

$$(I-P)\big(\nabla_x V_{\mathrm{q}}(\theta,x)\cdot\nabla_p H_0(X)\big) \neq 0 \quad \text{on } \big\{X\in\mathbb{R}^{2d},\ H_0(X)\in A\big\}$$

then, $b^*(E) > 0$ almost everywhere.

Proof. Regularity is a consequence of Corollary 4.4. Next, let $\varphi \in C^\infty_c(\mathbb{R})$. Thanks to Lemma 2.1-(iii), we get

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} b^*(H_0(X))\varphi^2(H_0(X))\, dX \\
&= \int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} \big(\nabla_x V_{\mathrm{q}}(\theta,x)\cdot\nabla_p H_0(X)\,\varphi(H_0(X))\big)\big(\chi^*_{\mathrm{q}}(\theta,X)\varphi(H_0(X))\big)\, d\theta\, dX \\
&= \int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} \mathcal{T}^*\big(\chi^*_{\mathrm{q}}(\theta,X)\varphi(H_0(X))\big)\,\chi^*_{\mathrm{q}}(\theta,X)\varphi(H_0(X))\, d\theta\, dX \\
&= \gamma\int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} \big|P\chi^*_{\mathrm{q}}(\theta,X)\varphi(H_0(X)) - \chi^*_{\mathrm{q}}(\theta,X)\varphi(H_0(X))\big|^2\, d\theta\, dX \geq 0.
\end{aligned}$$

Next, suppose that $b^*(E) = 0$ for E in some measurable set $A \subset \mathbb{R}$. Let us set

$$\chi^*_{\mathrm{q,A}}(\theta,X) = \chi^*_{\mathrm{q}}(\theta,X)\ \mathbb{1}_{\{X\in\mathbb{R}^{2d},\ H_0(X)\in A\}}(X).$$

Reasoning as above we obtain

$$\int_{\{X\in\mathbb{R}^{2d},\ H_0(X)\in A\}} b^*(H_0(X))\, dX = 0 = \gamma\int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} \big|P\chi^*_{\mathrm{q,A}} - \chi^*_{\mathrm{q,A}}\big|^2\, d\theta\, dX.$$

Therefore, $P\chi^\star_{\mathrm{q,A}} = \chi^*_{\mathrm{q,A}}$, which implies that

$$\begin{aligned}
\mathbb{1}_{\{X\in\mathbb{R}^{2d},\ H_0(X)\in A\}}(X)\ \mathcal{T}^*\chi^*_{\mathrm{q}} &= \mathcal{T}^*\chi^*_{\mathrm{q,A}} \\
&= \omega\cdot\nabla_\theta\chi^*_{\mathrm{q,A}} \\
&= P\big(\omega\cdot\nabla_\theta\chi^*_{\mathrm{q,A}}\big) \\
&= \mathbb{1}_{\{X\in\mathbb{R}^{2d},\ H_0(X)\in A\}}(X)\ \nabla_x V_{\mathrm{q}}(\theta,x)\cdot\nabla_p H_0(X)
\end{aligned}$$

holds. This would contradict the assumption $(I-P)(\mathbb{1}_{\{X\in\mathbb{R}^{2d},\ H_0(X)\in A\}}(X)$ $\nabla_x V_{\mathrm{q}}(\theta,x)\cdot\nabla_p H_0(X)) \neq 0$ and proves that $b^*(E) > 0$ for $E \in \mathbb{R}$ a.e. ∎

We end the proof by showing that (5.3) is a weak formulation of the conservative equation (1.7). The coarea formula yields

$$\frac{d}{dt}\int_{\mathbb{R}} F(t,E)\varphi(E)\,h_0(E)\,dE$$
$$= \int_{\mathbb{R}} F(t,E)\big(a^*(E)\partial_E\varphi(E) + b^*(E)\partial^2_{EE}\varphi(E)\big)\,h_0(E)\,dE,$$

with $F \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}, h_0(E)dE))$, and the right hand side makes sense by Lemma 5.4. Then, Lemma 3.1 allows us to write:

$$h_0(E)b^*(E) = h_0(E)b(E) \in L^2_{\mathrm{loc}}(\mathbb{R},\ h_0(E)^{-1}\,dE),$$
$$h_0(E)a^*(E) = \partial_E(h_0(E)b(E)) \in L^2_{\mathrm{loc}}(\mathbb{R},\ h_0(E)^{-1}\,dE).$$

Therefore, the right hand side in (5.3) becomes

$$\int_{\mathbb{R}} F(t,E)\ \partial_E(h_0(E)b(E)\partial_E\varphi(E))\ dE, \tag{5.4}$$

which proves the expected result. ∎

A.1. THE COAREA FORMULA AND ITS CONSEQUENCES

Let $H_0 : \mathbb{R}^{2d} \longrightarrow \mathbb{R}$ be a C^∞ function. The Sard Theorem (see Ref. (28)) asserts that, for almost every real number[6] $E \in \mathbb{R}$, and for any X such that $H_0(X) = E$, one has $\nabla_X H_0(X) \neq 0$. As a consequence, for almost every $E \in \mathbb{R}$, the level set $S_E := \{X \in \mathbb{R}^{2d},\ H_0(X) = E\}$ is a smooth, codimension one, submanifold of $\mathbb{R}^{2d}$. Now, the coarea formula asserts that the following equality holds

$$\int_{\mathbb{R}^{2d}} f(X)dX = \int_{\mathbb{R}}\left(\int_{S_E} f(X)\,\delta(H_0(X)-E)\right)dE, \tag{A.1}$$

for any function $f \in L^1(\mathbb{R}^{2d})$. We recall that the measure $\delta(H_0(X)-E)$ is defined by

$$\int_{S_E} f(X)\,\delta(H_0(X)-E) := \int_{S_E} f(X)\,\frac{d\sigma_E(X)}{\left|\nabla_X H_0(X)\right|}, \tag{A.2}$$

using again the fact that the gradient $\nabla_X H_0(X)$ never vanishes on S_E, $d\sigma_E(X)$ being the euclidian surface measure on the level set S_E. We recall that a crucial

[6] Note that here, we make the same abuse of notation as in the main part of the present paper: instead of writing the correct condition $E \in H_0(\mathbb{R}^{2d})$, we simply write $E \in \mathbb{R}$.

hypothesis in our work is

$$h_0(E) := \int_{S_E} \delta(H_0(X) - E) < \infty \tag{A.3}$$

for almost every $E \in \mathbb{R}$. Having defined the normalized average

$$\Pi f(E) = \frac{1}{h_0(E)} \int_{S_E} f(X)\delta(H_0(X) - E), \tag{A.4}$$

for $f \in L^1(\mathbb{R}^{2d})$, the coarea formula then takes the form

$$\int_{\mathbb{R}^{2d}} f(X)\, dX = \int_{\mathbb{R}} \Pi f(E)\, h_0(E)\, dE. \tag{A.5}$$

In particular, Π is an isometry from $L^1(\mathbb{R}^{2d})$ to $L^1(\mathbb{R}; h_0(E)\, dE)$. Since the analysis developed in the present paper needs an L^2 framework, we next turn to the L^2 properties of the operator Π.

Lemma A.1.1. *Let $f(X) : \mathbb{R}^{2d} \longrightarrow \mathbb{R}$ be in $L^2(\mathbb{R}^{2d})$. Then, we have*

$$\|\Pi f\|_{L^2(\mathbb{R}; h_0(E)dE)} \leq \|f\|_{L^2(\mathbb{R}^{2d})}.$$

Furthermore, let $g : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy $g \in L^2(\mathbb{R}; h_0(E)dE)$. The adjoint Π^* of the operator Π with respect to the scalar product in $L^2(\mathbb{R}; h_0(E)dE)$ is

$$\Pi^* g(X) = g(H_0(X)).$$

It satisfies

$$\|\Pi^* g\|_{L^2(\mathbb{R}^{2d})} = \|g\|_{L^2(h_0(E)dE)}.$$

Proof. First we use the Cauchy-Schwarz inequality together with the coarea formula and we get

$$\begin{aligned}
&\int_{\mathbb{R}} \left|\Pi f(E)\right|^2 h_0(E)\, dE \\
&\quad = \int_{\mathbb{R}} h_0(E) \left(\int_{S_E} f(X) \frac{\delta(H_0(X) - E)}{h_0(E)} \right)^2 dE \\
&\quad \leq \int_{\mathbb{R}} \frac{h_0(E)}{h_0(E)^2} \left(\int_{S_E} \left|f(X)\right|^2 \delta(H_0(X) - E) \right) \left(\int_{S_E} \delta(H_0(X) - E) \right) dE \\
&\quad \leq \int_{\mathbb{R}} \int_{S_E} \left|f(X)\right|^2 \delta(H_0(X) - E)\, dE = \int_{\mathbb{R}^{2d}} \left|f(X)\right|^2 dX.
\end{aligned}$$

Next, we observe that

$$\begin{aligned}
&\langle \Pi f, g \rangle_{L^2(h_0(E)dE)} \\
&\quad = \int_{\mathbb{R}} g(E) \left(\int_{S_E} f(X)\, \delta(H_0(X) - E) \right) dE \\
&\quad = \int_{\mathbb{R}} \int_{S_E} f(X)\, g(H_0(X))\, \delta(H_0(X) - E)\, dE = \int_{\mathbb{R}^{2d}} f(X)\, g(H_0(X))\, dX.
\end{aligned}$$

Eventually, the coarea formula yields

$$\begin{aligned}
\|\Pi^* g\|^2_{L^2(\mathbb{R}^{2d})} &= \int_{\mathbb{R}^{2d}} |g(H_0(X))|^2\, dX = \int_{\mathbb{R}} \int_{S_E} |g(H_0(X))|^2\, \delta(H_0(X) - E)\, dE \\
&= \int_{\mathbb{R}} |g(E)|^2 h_0(E)\, dE.
\end{aligned}$$

∎

B.1. DIMENSIONLESS EQUATIONS

Let us detail the passage from (1.1) to its dimensionless version (1.2). The coefficients of the operator Q being dimensionless, $Q(f)$ has the same dimension as f itself, while $\tau > 0$ is a relaxation time. Let us introduce time and length scales, denoted by T and L respectively, and let P stand for a momentum unit. Then, we set

$$\begin{cases} t_* = t/\mathrm{T}, \quad x_* = x/\mathrm{L}, \quad p_* = p/\mathrm{P}, \\ f_*(t_*, x_*, p_*) = \mathrm{L}^d \mathrm{P}^d\ f(t_* \mathrm{T}, x_* \mathrm{L}, p_* \mathrm{P}),\ H_{0,*}(x_*, p_*) = \dfrac{1}{\mathrm{H}}\ H_0(x_* \mathrm{L}, p_* \mathrm{P}), \end{cases}$$

where the energy scale $\mathrm{H} > 0$ characterizes the amplitude of the hamiltonian H_0. It remains to discuss the perturbation $\mathcal{V}$. To this end, we introduce additional parameters:

- $\varepsilon > 0$, which is a dimensionless quantity measuring the strength of the perturbation compared with the free hamiltonian,
- $\theta > 0$, which is a characteristic time scale of the evolution of $\mathcal{V}$.

Hence, we have

$$\mathcal{V}(t, x) = \varepsilon\ \mathrm{H}\ V_*\left(\frac{t}{\theta}, \frac{x}{\mathrm{L}}\right).$$

Finally, (1.1) can be recast in the following dimensionless form

$$\partial_{t_*} f_* + \frac{\mathrm{TH}}{\mathrm{LP}} \{H_{0,*}, f_*\} + \varepsilon\ \frac{\mathrm{TH}}{\mathrm{LP}} \{V_*(t_* \mathrm{T}/\theta), f_*\} = \frac{\mathrm{T}}{\tau}\ Q_*(f_*).$$

Then, our analysis is based on the following scaling assumptions. First, we suppose that

$$\frac{\mathrm{TH}}{\mathrm{LP}} = \frac{1}{\varepsilon^2} \gg 1.$$

Roughly speaking it means that the time unit we adopt is large compared with the characteristic time scale of the free hamiltonian H_0 (e.g. for the harmonic oscillator the period of the characteristic curves). Next, we are interested in the behavior of the system as $\varepsilon \ll 1$ when the time scales involved in the problem satisfy the following ordering:

$$\frac{\mathrm{T}}{\theta} = \frac{1}{\varepsilon^2}, \qquad \frac{\mathrm{T}}{\tau} = \frac{\gamma}{\varepsilon^2}, \qquad \gamma = \mathcal{O}(1).$$

Here, $\gamma > 0$ is a fixed dimensionless quantity. This sets up the asymptotic regime we are dealing with.

C.1. EFFECTIVE COEFFICIENTS: PROOF OF LEMMA 3.1

Let $\psi \in C_c^\infty(\mathbb{R})$. The coarea formula (A.5) yields

$$\begin{aligned}
\int_{\mathbb{R}} h_0 b^*\ \psi\, dE &= \int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} \left\{V_{\mathrm{q}}, H_0\right\}\ \chi_{\mathrm{q}}^*(\theta, X)\ \psi(H_0(X))\, d\theta\, dX \\
&= \int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} \mathcal{T}\chi_{\mathrm{q}}\ \chi_{\mathrm{q}}^*(\theta, X)\ \psi(H_0(X))\, d\theta\, dX \\
&= \int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} \chi_{\mathrm{q}}\ \mathcal{T}^*\big(\chi_{\mathrm{q}}^*(\theta, X)\ \psi(H_0(X))\big)\, d\theta\, dX \\
&= \int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} \chi_{\mathrm{q}}\ \mathcal{T}^*\chi_{\mathrm{q}}^*(\theta, X)\ \psi(H_0(X))\, d\theta\, dX \\
&= \int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} \chi_{\mathrm{q}}\ \left\{V_{\mathrm{q}}, H_0\right\}\ \psi(H_0(X))\, d\theta\, dX = \int_{\mathbb{R}} h_0 b\ \psi\, dE.
\end{aligned}$$

Similarly, combining the coarea formula and integration by parts, we get

$$\begin{aligned}
\int_{\mathbb{R}} h_0 a^*\ \psi\, dE &= \int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} \left\{V_{\mathrm{q}}, \chi_{\mathrm{q}}^*\right\}(\theta, X)\ \psi(H_0(X))\, d\theta\, dX \\
&= -\int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} \chi_{\mathrm{q}}^*\ \left\{V_{\mathrm{q}}, \psi(H_0(X))\right\}\, d\theta\, dX \\
&= -\int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} \chi_{\mathrm{q}}^*\ \left\{V_{\mathrm{q}}, H_0(X)\right\}\ (\partial_E\psi)(H_0(X))\, d\theta\, dX \\
&= -\int_{\mathbb{R}} h_0 b^*\ \partial_E\psi\, dE,
\end{aligned}$$

which proves $h_0 a^\star = \partial_E(h_0 b^*)$. We obtain the equality $h_0 a^* = h_0 a$ by remarking that

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} \chi_q^* \left\{V_q, H_0(X)\right\} (\partial_E \psi)(H_0(X))\, d\theta\, dX \\
&\quad = \int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} \chi_q^*\, \mathcal{T} \chi_q\, (\partial_E \psi)(H_0(X))\, d\theta\, dX \\
&\quad = \int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} \mathcal{T}^* \chi_q^*\, \chi_q\, (\partial_E \psi)(H_0(X))\, d\theta\, dX \\
&\quad = \int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} \left\{V_q, H_0(X)\right\} \chi_q\, (\partial_E \psi)(H_0(X))\, d\theta\, dX \\
&\quad = \int_{\mathbb{R}^{2d}}\int_{\mathbb{Y}} \left\{V_q, \psi(H_0(X))\right\} \chi_q\, d\theta\, dX
\end{aligned}$$

holds. An integration by parts allows to conclude the proof. ∎

D.1. DOUBLE SCALE CONVERGENCE: PROOF OF PROPOSITION 5.2

The double scale convergence framework has been extended to very complicated and general oscillating coefficients, which leads to tedious technicalities; we refer on these aspects to Ref. (11, 30). The case of quasi-periodic coefficients we are dealing with can be treated by following closely the arguments of Ref. (3). Indeed, consider a bounded sequence in $L^2(\mathbb{R})$

$$\sup_{\varepsilon>0} \int_{\mathbb{R}} |f_\varepsilon(t)|^2\, dt \le C < \infty.$$

Let $\mathcal{A}$ stand for the space $L^2(\mathbb{R}; C^0_\#(\mathbb{Y}))$, which is a separable Banach space. Let $\phi \in \mathcal{A}$ and remark that

$$\begin{aligned}
\left| \int_{\mathbb{R}} f_\varepsilon(t)\, \phi(t, \omega t/\varepsilon)\, dt \right| &\le \|f_\varepsilon\|_{L^2(\mathbb{R})} \left(\int_{\mathbb{R}} \left(\sup_{z\in Y} |\phi(t,z)| \right)^2 dt \right)^{1/2} \\
&\le C\, \|\phi\|_{\mathcal{A}}.
\end{aligned}$$

Hence, if we denote by Θ_ε the linear form defined by

$$\langle \Theta_\varepsilon, \phi \rangle = \int_{\mathbb{R}} f_\varepsilon(t)\, \phi(t, \omega t/\varepsilon)\, dt,$$

we conclude that $(\Theta_\varepsilon)_{\varepsilon>0}$ is bounded in the dual set $\mathcal{A}'$. Hence, by the Banach-Alaoglu theorem, we can suppose that Θ_ε converges to some ν weakly-$*$ in $\mathcal{A}'$.

However, we also have:

$$\left|\int_{\mathbb{R}} f_\varepsilon(t)\,\phi(t,\omega t/\varepsilon)\,dt\right| \leq C\,\left(\int_{\mathbb{R}} |\phi(t,\omega t/\varepsilon)|^2\,dt\right)^{1/2},$$

so that letting ε tend to 0 yields:

$$|\langle \nu,\phi\rangle| \leq C\,\lim_{\varepsilon\to 0}\left(\int_{\mathbb{R}} |\phi(t,\omega t/\varepsilon)|^2\,dt\right)^{1/2}.$$

Therefore, we can identify ν with a function $F \in L^2_\#(\mathbb{R}\times\mathbb{Y})$ by the Riesz theorem once we are able to justify that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} |\phi(t,\omega t/\varepsilon)|^2\,dt = \int_{\mathbb{Y}}\int_{\mathbb{R}} |\phi(t,\theta)|^2\,d\theta\,dt.$$

The proof of this fact follows the arguments of Ref. (3), with some slight modifications; the adaptation to the quasi-periodic framework can be seen as a version of the Birkhoff ergodic theorem, see Ref. (15). It is a consequence of the two following claims.

Lemma D.1.1. *Let ω be a element of $\mathbb{R}^r$ the components of which are rationaly independent. Let $\phi \in C^0_\#(\mathbb{Y})$. Then $\phi(\omega t/\varepsilon) \rightharpoonup \int_{\mathbb{Y}} \phi(\theta)\,d\theta$ weakly-$*$ in $L^\infty(\mathbb{R})$.*

Proof. We start by proving the result for $\phi(\theta) = \exp(2i\pi k\cdot\theta)$, $k \in \mathbb{Z}^r$. Indeed, let $\psi \in L^1(\mathbb{R})$. We get

$$\int_{\mathbb{R}} \psi(t) e^{2i\pi k\cdot\omega t/\varepsilon}\,dt\widehat{\psi}\left(-\frac{2\pi k\cdot\omega}{\varepsilon}\right).$$

Therefore, for $k = 0$, this is nothing but

$$\widehat{\psi}(0) = \int_{\mathbb{R}} \psi(t)\,dt = \int_{\mathbb{R}} \psi(t)\,dt \int_{\mathbb{Y}} e^{2i\pi 0\cdot\theta}\,d\theta,$$

while for $k \neq 0$, the ergodic condition $k\cdot\omega \neq 0$ yields

$$\lim_{\varepsilon\to 0}\widehat{\psi}\left(-\frac{2\pi k\cdot\omega}{\varepsilon}\right) = 0 = \int_{\mathbb{R}} \psi(t)\,dt \int_{\mathbb{Y}} e^{2i\pi k\cdot\theta}\,d\theta.$$

Of course, we immediately deduce that the result also applies to any trigonometric polynomial.

Then, we extend the property to any $\phi \in C^0_\#(\mathbb{Y})$. Indeed, such a function can be approached, in the sup norm sense, by a sequence $(p_n)_{n\in\mathbb{N}}$ of trigonometric polynomials. Then, we note that

$$\left| \int_{\mathbb{R}} \psi(t)\phi(\omega t/\varepsilon)\,dt - \int_{\mathbb{R}} \psi(t) \left(\int_{\mathbb{Y}} \phi(\theta)\,d\theta \right) dt \right|$$

$$\leq \int_{\mathbb{R}} |\psi(t)|\, |\phi(\omega t/\varepsilon) - p_n(\omega t/\varepsilon)|\,dt + \left| \int_{\mathbb{R}} \psi(t) p_n(\omega t/\varepsilon)\,dt \right.$$

$$\left. - \int_{\mathbb{R}} \psi(t) \left(\int_{\mathbb{Y}} p_n(\theta)\,d\theta \right) dt \right| + \int_{\mathbb{R}} |\psi(t)| \int_{\mathbb{Y}} |\phi(\theta) - p_n(\theta)|\,d\theta\,dt$$

$$\leq 2\|\psi\|_{L^1(\mathbb{R})} \|\phi - p_n\|_{L^\infty(\mathbb{Y})} + \left| \int_{\mathbb{R}} \psi(t) p_n(\omega t/\varepsilon)\,dt - \int_{\mathbb{R}} \psi(t) \left(\int_{\mathbb{Y}} p_n(\theta)\,d\theta \right) dt \right|.$$

Let $\delta > 0$ be a positive number. Then, there exists $n = n(\delta)$ such that the first term at the right hand side is less than δ. Eventually, the previous step of the proof garantees that for $0 < \varepsilon < \varepsilon(\delta)$ small enough, the last integral is also less than δ. This ends the proof. ∎

Lemma D.1.2. *Let ω be an element of $\mathbb{R}^r$ the components of which are rationaly independent. Let $\phi \in L^1(\mathbb{R}; C^0_\#(\mathbb{Y}))$. Then, we have*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} \phi(t, \omega t/\varepsilon)\,dt = \int_{\mathbb{Y}} \int_{\mathbb{R}} \phi(t, \theta)\,d\theta\,dt.$$

Proof. Let us introduce a covering of the unit cube of $\mathbb{R}^r$, made of $I(n)$ open sets O_i with diameter $\leq \alpha_n$, where we assume that $I(n) \to \infty$ and $\alpha_n \to 0$ as n goes to ∞. For each $i \in \{1, \ldots, I(n)\}$, Let θ_i be an element of O_i. To this covering, we associate a set of functions ζ_i, $i \in \{1, \ldots, I(n)\}$ such that

$$0 \leq \zeta_i(\theta) \leq 1, \quad \operatorname{supp}(\zeta_i) \subset O_i, \quad \sum_{i=1}^{I(n)} \zeta_i(\theta) = 1,$$

and we extend these functions to $\mathbb{R}^r$ by periodicity. Let $\phi \in L^1(\mathbb{R}; C^0_\#(\mathbb{Y}))$. We set

$$\phi_n(t, \theta) = \sum_{i=1}^{I(n)} \phi(t, \theta_i)\, \zeta_i(\theta).$$

Then, we note that

$$\begin{aligned}\left|\phi(t,\theta)-\phi_n(t,\theta)\right| &= \left|\sum_{i=1}^{I(n)} \zeta_i(\theta)\,(\phi(t,\theta_i)-\phi(t,\theta))\right| \\ &\le \sum_{i=1}^{I(n)} \zeta_i(\theta)\ \sup_{\theta\in O_i}\left|\phi(t,\theta_i)-\phi(t,\theta)\right|.\end{aligned}$$

Since, for $t\in\mathbb{R}$ a.e., the function $\theta\mapsto\phi(t,\theta)$ is continuous on the compact set $\mathbb{Y}$, and for $\theta\in O_i$, $|\theta-\theta_i|\le\alpha_n\to 0$, we deduce that $\sup_{\theta\in\mathbb{Y}}\left|\phi(t,\theta)-\phi_n(t,\theta)\right|\to 0$ as n goes to ∞. Besides, we have $\sup_{\theta\in\mathbb{Y}}\left|\phi(t,\theta)-\phi_n(t,\theta)\right|\le 2\,\|\phi(t,\cdot)\|_{L^\infty(\mathbb{Y})}\in L^1(\mathbb{R})$. Therefore, the Lebesgue theorem yields

$$\|\phi-\phi_n\|_{L^1(\mathbb{R},L^\infty(\mathbb{Y}))}\underset{n\to\infty}{\longrightarrow}0. \tag{D.1}$$

Then, for $n\in\mathbb{N}$ fixed, we write

$$\int_{\mathbb{R}}\phi_n(t,\omega t/\varepsilon)\,dt=\sum_{i=1}^{I(n)}\int_{\mathbb{R}}\phi(t,\theta_i)\,\zeta_i(\omega t/\varepsilon)\,dt.$$

Since $t\mapsto\phi(t,\theta_i)$ belongs to $L^1(\mathbb{R})$ and $\zeta_i\in C^0_\#(\mathbb{Y})$, Lemma D.1.1 applies and leads to

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\phi_n(t,\omega t/\varepsilon)\,dt=\sum_{i=1}^{I(n)}\int_{\mathbb{R}}\phi(t,\theta_i)\left(\int_{\mathbb{Y}}\zeta_i(\theta)\,d\theta\right)\,dt=\int_{\mathbb{R}}\int_{\mathbb{Y}}\phi_n(t,\theta)\,d\theta\,dt.$$

Combining this to (D.1) ends the proof. ∎

E.1. A SIMPLE EXAMPLE

It is worth illustrating the previous developments with a fully explicit computation. This can be performed when considering Hamiltonians based on the harmonic oscillator

$$H_{\rm harm}(X)=|X|^2/2=\frac{x^2+p^2}{2},$$

with $X=(x,p)\in\mathbb{R}^2$ and the simplest perturbation

$$V(t/\varepsilon^2,x)=x\cos(\omega t/\varepsilon^2),\quad \omega\in\mathbb{R}\setminus\{0\}.$$

Let us consider the following Hamiltonian

$$H_0(X)=G(H_{\rm harm}(X)),$$

with $G : \mathbb{R}^+ \to \mathbb{R}^+$ a C^1, strictly increasing function. We note that $H_0(X) = E$ iff $|X|^2 = 2G^{-1}(E)$. Therefore, integration over S_E reduces to integration over the sphere of $\mathbb{R}^2$ with radius $2G^{-1}(E)$: we write $(x, p) \in S_E$ as $x = \sqrt{2G^{-1}(E)}\ \cos(\sigma)$, $p = \sqrt{2G^{-1}(E)}\ \sin(\sigma)$, with $\sigma \in (0, 2\pi)$ and $d\sigma_E$ becomes $\sqrt{2G^{-1}(E)}\, d\sigma$. Next, we compute

$$\nabla H_0(X) = G'(|X|^2/2) \begin{pmatrix} x \\ p \end{pmatrix},$$

so that $|\nabla H_0(X)| = G'(|X|^2/2)\ |X| = G'(G^{-1}(E))\ \sqrt{2G^{-1}(E)}$. In what follows, we denote

$$\Omega(E) = G'(G^{(-1)}(E)).$$

Hence, we obtain

$$h_0(E) = \int_{x^2+p^2=2G^{-1}(E)} \frac{d\sigma_E}{|\nabla H_0(x, p)|} \int_0^{2\pi} \frac{\sqrt{2G^{(-1)}(E)}}{\Omega(E)\ \sqrt{2G^{(-1)}(E)}}\, d\sigma = \frac{2\pi}{\Omega(E)},$$

and

$$\Pi f(E) = \frac{1}{2\pi} \int_0^{2\pi} f\Big(\sqrt{2G^{(-1)}(E)}\cos(\sigma), \sqrt{2G^{(-1)}(E)}\sin(\sigma)\Big)\, d\sigma.$$

The characteristics $\overline{X}(t; x, p) = \big(\overline{x}(t; x, p), \overline{p}(t; x, p)\big)$ verify

$$\frac{d}{dt}\overline{X}(t; x, p) = G'(|\overline{X}(t; x, p)|^2/2) \begin{pmatrix} \overline{p}(t; x, p) \\ -\overline{x}(t; x, p) \end{pmatrix}, \qquad \overline{X}(0; x, p) = \begin{pmatrix} x \\ p \end{pmatrix}.$$

The keypoint relies on the observation that $X(t; x, p)$ lies on the same sphere of $\mathbb{R}^2$ than the initial data. Indeed, we have

$$\frac{d}{dt} H_0\big(\overline{X}(t; x, p)\big) = 0.$$

Since G is a diffeomorphism, we deduce that

$$\overline{x}(t; x, p)^2 + \overline{p}(t; x, p)^2 = x^2 + p^2 = 2G^{-1}(E).$$

In turn, $\overline{x}(t; x, p)$ satisfies the following simple second order ODE

$$\begin{aligned} \frac{d^2}{dt^2}\overline{x}(t; x, p) &= \frac{d}{dt}\big[G'(|\overline{X}(t; x, p)|^2/2)\ \overline{p}(t; x, p)\big] G'(|\overline{X}(t; x, p)|^2/2)\frac{d}{dt}\overline{p}(t; x, p) \\ &= -\Omega(E)^2\ \overline{x}(t; x, p). \end{aligned}$$

We immediately solve this ODE, and we finally obtain

$$\overline{X}(t; x, p) = \begin{pmatrix} \cos(\Omega(E)t) & \sin(\Omega(E)t) \\ -\sin(\Omega(E)t) & \cos(\Omega(E)t) \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}, \quad E = G\Big(\frac{x^2 + p^2}{2}\Big).$$

In particular, we note that

$$\nabla_{x,p}\overline{X}(t;x,p) = \begin{pmatrix} \cos(\Omega(E)t) + \overline{p}(t;x,p)\, t\Omega\Omega'(E)\, x & \sin(\Omega(E)t) + \overline{p}(t;x,p)\, t\Omega\Omega'(E)\, p \\ -\sin(\Omega(E)t) - \overline{x}(t;x,p)\, t\Omega\Omega'(E)\, x & \cos(\Omega(E)t) - \overline{x}(t;x,p)\, t\Omega\Omega'(E)\, p \end{pmatrix}.$$

Therefore, Hypothesis 1.3 is satisfied since $E \mapsto \Omega\Omega'(E)$ is locally bounded. Of course, this is also true in the purely harmonic case ($G(h) = h$, $\Omega(E) = 1$).
It remains to compute the effective coefficients. Since $\partial_p H_0(x,p) = \Omega(E)\, p$, we get

$$\chi(\theta,x,p) \int_0^\infty e^{-\gamma s} \cos(\theta - \omega s)\, \Omega(E)(x \sin(\Omega(E)s) + p\cos(\Omega(E)s))\, ds.$$

Then, we are led to

$$\begin{aligned} b(E) &= \Pi\left(\int_0^{2\pi} \partial_x V(\theta,x)\partial_p H_{\text{harm}}\, \chi^*(\theta,x,p)\, d\theta\right)(E) \\ &= \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi}\int_0^{2\pi} \cos(\theta)\, \Omega(E)\sqrt{2G^{-1}(E)}\sin(\sigma)\, e^{-\gamma s}\cos(\theta - \omega s) \\ &\quad \times \Omega(E)\sqrt{2G^{-1}(E)}\big(\cos(\sigma)\sin(\Omega(E)s) + \sin(\sigma)\cos(\Omega(E)s)\big)\, d\theta\, d\sigma\, ds \\ &= \frac{2G^{-1}(E)\,\Omega(E)^2}{2\pi}\int_0^\infty e^{-\gamma s}\,\pi\, \cos(\Omega(E)s)\left(\int_0^{2\pi}\cos(\theta)\cos(\theta-\omega s)\, d\theta\right) ds \\ &= \pi G^{-1}(E)\Omega(E)^2 \int_0^\infty e^{-\gamma s}\cos(\Omega(E)s)\cos(\omega s)\, ds \\ &= \pi\frac{G^{-1}(E)\Omega(E)^2}{2}\left(\frac{\gamma}{(\omega+\Omega(E))^2+\gamma^2} + \frac{\gamma}{(\omega-\Omega(E))^2+\gamma^2}\right). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} a(E) &= \Pi\left(\int_0^{2\pi} \partial_x V(\theta,x)\partial_p \chi^*(\theta,x,p)\, d\theta\right)(E) \\ &= \frac{1}{h_0(E)}\partial_E(h_0 b^*(E)) \\ &= \frac{\pi}{2}\Omega(E)\partial_E\left[\Omega(E)G^{-1}(E)\left(\frac{\gamma}{(\omega+\Omega(E))^2+\gamma^2} + \frac{\gamma}{(\omega-\Omega(E))^2+\gamma^2}\right)\right] \end{aligned}$$

Let us end with a couple of remarks concerning these computations. Notice that the diffusion coefficient $b(E)$ vanishes when $G^{-1}(E)$ or $\Omega(E)$ vanish, which is the case for the harmonic oscillator at the energy $E = 0$. The coefficient becomes infinite when G has an infinite derivative. Remark that the limit $\gamma \to 0$ reveals resonance phenomena: dealing with the purely harmonic case ($G(h) = h$, $\Omega(E) = 1$), we remark that the coefficients tend to ∞ as $\gamma \to 0$ if the perturbation V oscillates with the characteristic frequency of the system $\omega = \pm 1$. The situation can be different when dealing with another function G. Indeed, f the equation $\Omega(E) = \pm\omega$ has a finite number of solutions $\{E_1, \dots, E_I\}$, resonances only occur on this finite set of energies.

Of course, it is also interesting to compare with the explicit solution of the kinetic equation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon^2}\{H_0, f^\varepsilon\} + \frac{1}{\varepsilon}\{V(t/\varepsilon^2), f^\varepsilon\} = 0,$$

that can be obtained in the simplest case $H_0(x, p) = (x^2 + p^2)/2$ and $V(t, x) = x\cos(\omega t)$. Indeed, the characteristics associated with the full Hamiltonian can be readily computed. They satisfy the ODE system

$$\begin{cases} \dfrac{d}{ds}\widetilde{x}(s;t,x,p) = \dfrac{1}{\varepsilon^2}\widetilde{p}(s;t,x,p), & \dfrac{d}{ds}\widetilde{p}(s;t,x,p) = -\dfrac{1}{\varepsilon^2}\widetilde{x}(s;t,x,p) \\ & \qquad + \dfrac{1}{\varepsilon}\cos(\omega s/\varepsilon^2), \\ \widetilde{x}(t;t,x,p) = x, & \widetilde{p}(t;t,x,p) = p. \end{cases}$$

We get for $\omega \neq \pm 1$:

$$\begin{aligned} \widetilde{x}(0;t,x,p) &= x\cos(t/\varepsilon^2) - p\sin(t/\varepsilon^2) \\ &\quad + \frac{\varepsilon}{2}\left(\frac{1-\cos((1+\omega)t/\varepsilon^2)}{1+\omega} + \frac{1-\cos((1-\omega)t/\varepsilon^2)}{1-\omega}\right), \\ \widetilde{p}(0;t,x,p) &= x\sin(t/\varepsilon^2) + p\cos(t/\varepsilon^2) \\ &\quad - \frac{\varepsilon}{2}\left(\frac{\sin((1+\omega)t/\varepsilon^2)}{1+\omega} + \frac{\sin((1-\omega)t/\varepsilon^2)}{1-\omega}\right), \end{aligned}$$

and for $\omega = \pm 1$:

$$\begin{aligned} \widetilde{x}(0;t,x,p) &= x\cos(t/\varepsilon^2) - p\sin(t/\varepsilon^2) + \frac{\varepsilon}{2}\,\frac{1-\cos(2t/\varepsilon^2)}{2}, \\ \widetilde{p}(0;t,x,p) &= x\sin(t/\varepsilon^2) + p\cos(t/\varepsilon^2) - \frac{\varepsilon}{4}\sin(2t/\varepsilon^2) - \frac{t}{2\varepsilon}. \end{aligned}$$

Given an initial data f_0, we thus have

$$f^\varepsilon(t,x,p) = f_0\big(\widetilde{x}(0;t,x,p), \widetilde{p}(0;t,x,p)\big),$$

which develops different features than solutions of a diffusion equation.

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